First-Order Logic

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Based on "Artificial Intelligence" by S. Russell and P. Norvig, 2003
Chapter 8

- Why FOL?
- Syntax and semantics of FOL
- Fun with sentences
- Wumpus world in FOL

First-order logic

- Whereas propositional logic assumes the world contains just **facts**, first-order logic assumes it also contains
  - **Objects**: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries, ...
  - **Relations**: red, round, bogus, prime, multi storied, ..., brother of, bigger than, inside, part of, has color, occured after, owns, comes between, ...
  - **Functions**: father of, best friend, third inning of, one more than, beginning of, ...

Syntax of FOL: Basic elements

Constants	$KingJohn,\ 2, \ldots$
Predicates	$Brother,\ >, \ldots$
Functions	$Sqrt,\ LeftLegOf, \ldots$
Variables	$x,\ y,\ a,\ b, \ldots$
Connectives	$\land\ \lor\ \neg\ \Rightarrow \Leftrightarrow$
Equality	$=$
Quantifiers	$\forall\ \exists$

Atomic sentences

Atomic sentence $=$ $predicate(term_1, \ldots, term_n)$
or $term_1 = term_2$

Term $=$ $function(term_1, \ldots, term_n)$
or *constant* or *variable*

E.g., $Brother(KingJohn, RichardTheLionheart)$

$> (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))$

Complex sentences

- Complex sentences are made from atomic sentences using connectives

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2$$

E.g. $Sibling(KingJohn, Richard) \Rightarrow Sibling(Richard, KingJohn)$

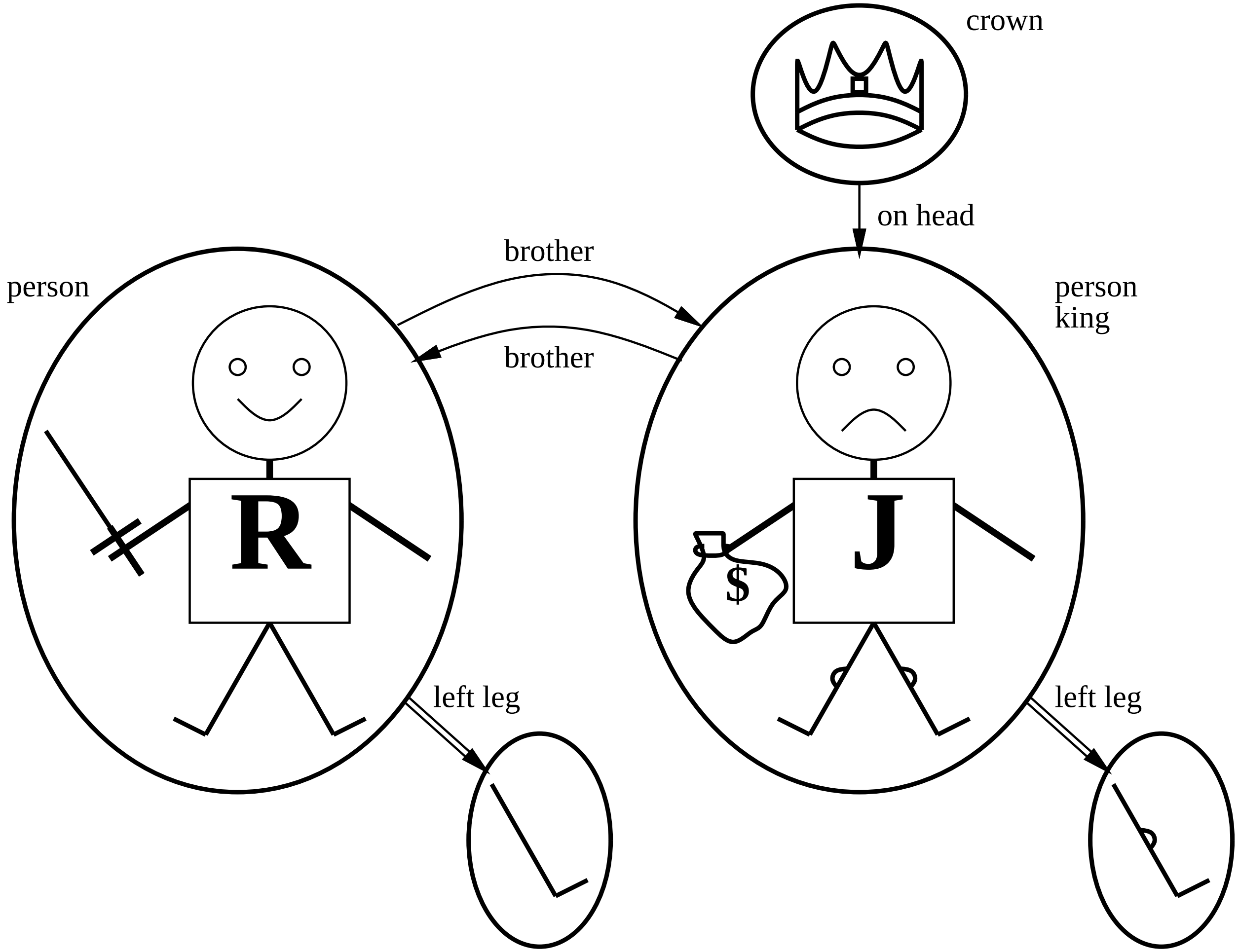
$>(1,2) \vee \leq(1,2)$

$>(1,2) \wedge \neg{>}(1,2)$

Truth in first-order logic

- Sentences are true with respect to a **model** and an **interpretation**
- Model contains objects and relations among them
- Interpretation specifies referents for
  - *constant symbols* → objects
  - *predicate symbols* → relations
  - *function symbols* → functional relations
- An atomic sentence $predicate(term_1, \ldots, term_n)$ is true iff the objects referred to by $term_1, \ldots, term_n$ are in the relation referred to by *predicate*

Models for FOL: Example



Universal quantification

- $\forall < variables > < sentence >$
- Everyone at Waikato is smart: $\forall x At(x, Waikato) \Rightarrow Smart(x)$
- Roughly speaking, $\forall x P$ is equivalent to the conjunction of instantiations of P

$$
\begin{array}{ll}
 & At(KingJohn, Waikato) \Rightarrow Smart(KingJohn) \\
\wedge & At(Richard, Waikato) \Rightarrow Smart(Richard) \\
\wedge & At(Waikato, Waikato) \Rightarrow Smart(Waikato) \\
\wedge & \ldots
\end{array}
$$

A common mistake to avoid

- Typically, $\Rightarrow$ is the main connective with $\forall$.
- Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall x At(x, Waikato) \wedge Smart(x)$$

means "Everyone is at Waikato and everyone is smart"

Existential quantification

- $\exists < variables > < sentence >$
- Someone at Otago is smart: $\exists x At(x, Otago) \land Smart(x)$
- Roughly speaking, $\exists x P$ is equivalent to the disjunction of instantiations of P

$$
\begin{array}{ll}
 & At(KingJohn, Otago) \land Smart(KingJohn) \\
\lor & At(Richard, Otago) \land Smart(Richard) \\
\lor & At(Otago, Otago) \land Smart(Otago) \\
\lor & \ldots
\end{array}
$$

Another common mistake to avoid

- Typically, $\wedge$ is the main connective with $\exists$.
- Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

$$\exists x At(x, Otago) \Rightarrow Smart(x)$$

is true if there is anyone who is not at Otago!

Properties of quantifiers

- $\forall x \forall y$ is the same as $\forall y \forall x$
- $\exists x \exists y$ is the same as $\exists y \exists x$
- $\exists x \forall y$ is not the same as $\forall y \exists x$
- $\exists x \forall y Loves(x, y)$
  "There is a person who loves everyone in the world"
- $\forall y \exists x Loves(x, y)$
  "Everyone in the world is loved by at least one person"
- Quantifier duality: each can be expressed using the other
  - $\forall x Likes(x, IceCream)$ $\qquad \neg \exists x \neg Likes(x, IceCream)$
  - $\exists x Likes(x, Broccoli)$ $\qquad \neg \forall x \neg Likes(x, Broccoli)$

Fun with sentences

- Brothers are siblings
  - ?
- “Sibling” is reflexive
  - ?
- One’s mother is one’s female parent
  - ?
- A first cousin is a child of a parent’s sibling
  - ?

Fun with sentences

- Brothers are siblings
  - $\forall x, y Brother(x, y) \Rightarrow Sibling(x, y)$
- "Sibling" is reflexive
  - $\forall x, y Sibling(x, y) \Leftrightarrow Sibling(y, x)$
- One's mother is one's female parent
  - $\forall x, y Mother(x, y) \Leftrightarrow (Female(x) \wedge Parent(x, y))$
- A first cousin is a child of a parent's sibling
  - $\forall x, y FirstCousin(x, y) \Leftrightarrow$
    $\exists p, ps Parent(p, x) \wedge Sibling(ps, p) \wedge Parent(ps, y)$

Equality

- $term_1 = term_2$ is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object
- E.g., definition of $Sibling$ in terms of $Parent$:
  $\forall x, y Sibling(x, y) \Leftrightarrow [\neg(x = y) \wedge \exists m, f \neg(m = f) \wedge Parent(m, x) \wedge Parent(f, x) \wedge Parent(m, y) \wedge Parent(f, y)]$

Interacting with FOL KBs

- Suppose a wumpus-world agent is using an FOL KB and perceives a smell and a breeze (but no glitter) at $t = 5$:
  - $\text{Tell}(KB, Percept([Smell, Breeze, Glitter], 5))$
  - $\text{Ask}(KB, \exists a\, Action(a, 5))$
- Answer: Yes, $\{a/Grab\}$ $\quad\leftarrow$ substitution (binding list)
- Given a sentence S and a substitution σ, $S\sigma$ denotes the result of plugging σ into S; e.g.,
  $S = Smarter(x, y)$
  $\sigma = \{x/Hillary, y/Bill\}$
  $S\sigma = Smarter(Hillary, Bill)$
- Ask(KB,S) returns some/all σ such that $KB \models S\sigma$

Knowledge base for the wumpus world

- "Perception"
  $\forall b, g, t Percept([Smell, b, g], t) \Rightarrow Smelled(t)$
  $\forall s, b, t Percept([s, b, Glitter], t) \Rightarrow AtGold(t)$
- Reflex: $\forall t AtGold(t) \Rightarrow Action(Grab, t)$
- Reflex with internal state: do we have the gold already?
  $\forall t AtGold(t) \land \neg Holding(Gold, t) \Rightarrow Action(Grab, t)$
- $Holding(Gold, t)$ cannot be observed
  $\Rightarrow$ keeping track of change is essential

Deducing hidden properties

- Properties of locations:
  $\forall l, t\, At(Agent, l, t) \land Smelled(t) \Rightarrow Smelly(l)$
  $\forall l, t\, At(Agent, l, t) \land Breeze(t) \Rightarrow Breezy(l)$
- Squares are breezy near a pit:
  - Diagnostic rule—infer cause from effect
    $\forall y\, Breezy(y) \Rightarrow \exists x\, Pit(x) \land Adjacent(x, y)$
  - Causal rule—infer effect from cause
    $\forall x, y\, Pit(x) \land Adjacent(x, y) \Rightarrow Breezy(y)$
  - Neither of these is complete—e.g., the causal rule doesn't say whether squares far away from pits can be breezy
  - Definition for the *Breezy* predicate:
    $\forall y\, Breezy(y) \Leftrightarrow [\exists x\, Pit(x) \land Adjacent(x, y)]$