

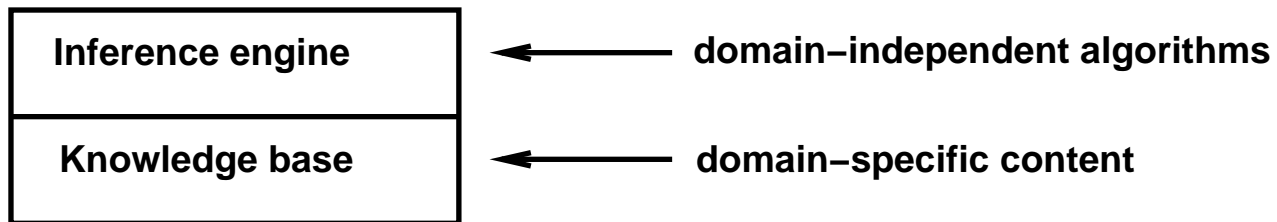
PROPOSITIONAL LOGIC

CHAPTER 7

Outline

- ◇ Knowledge-based systems
- ◇ Wumpus world
- ◇ Models and entailment
- ◇ Propositional logic
- ◇ Equivalence, validity, satisfiability
- ◇ Inference rules and theorem proving
 - forward chaining
 - backward chaining
 - resolution

Knowledge bases



Knowledge base = set of sentences in a **formal** language

Declarative approach to building an agent (or other system):

TELL it what it needs to know

Then it can **ASK** itself what to do—answers should follow from the KB

A (simplified) Wumpus world

Goal: avoid Wumpus and pits, find gold

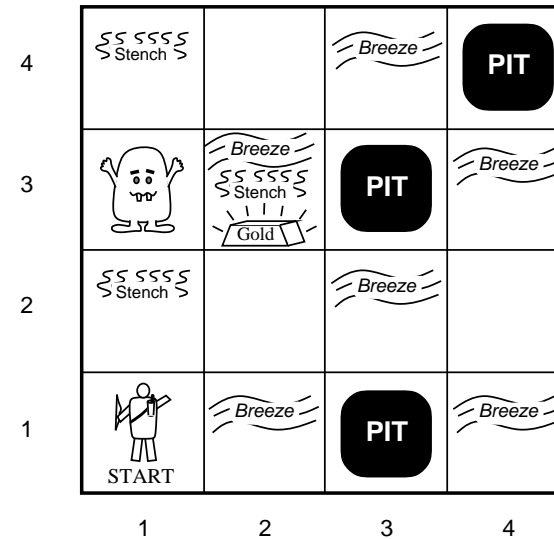
Environment:

Squares adjacent to wumpus are smelly

Squares adjacent to pit are breezy

Glitter iff gold is in the same square

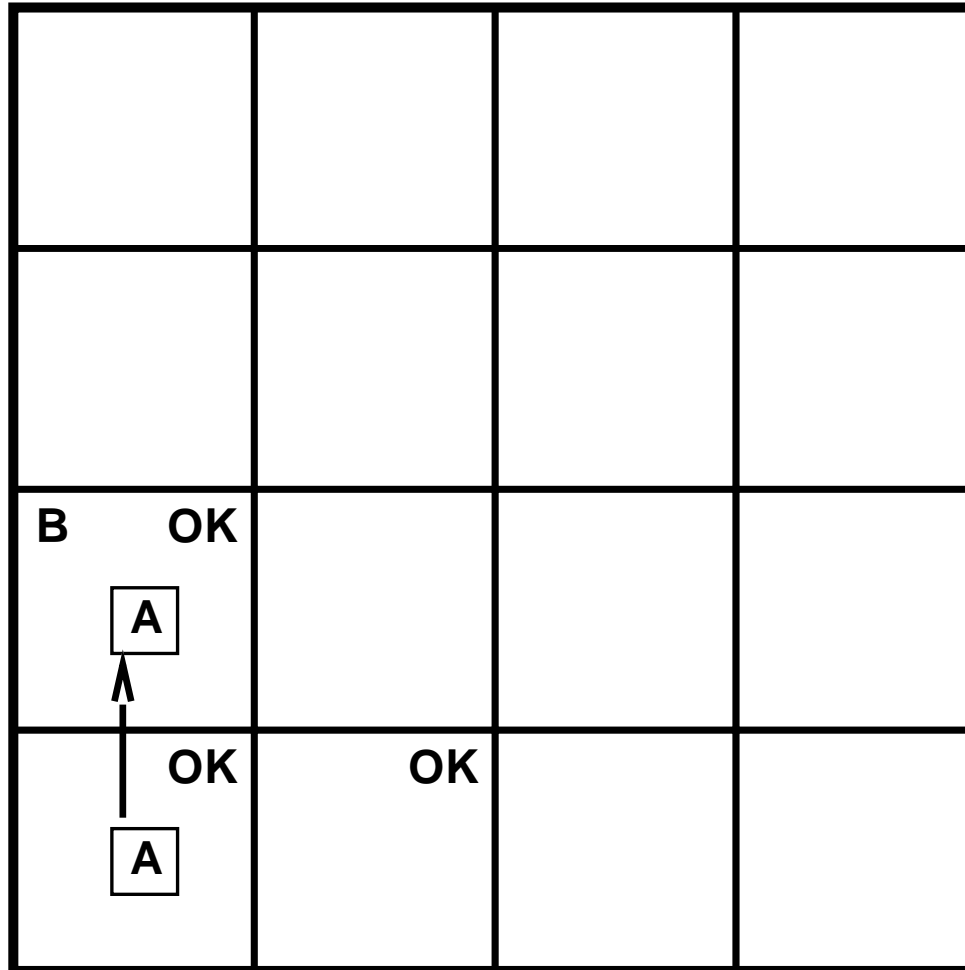
Sensors: Breeze, Glitter, Smell



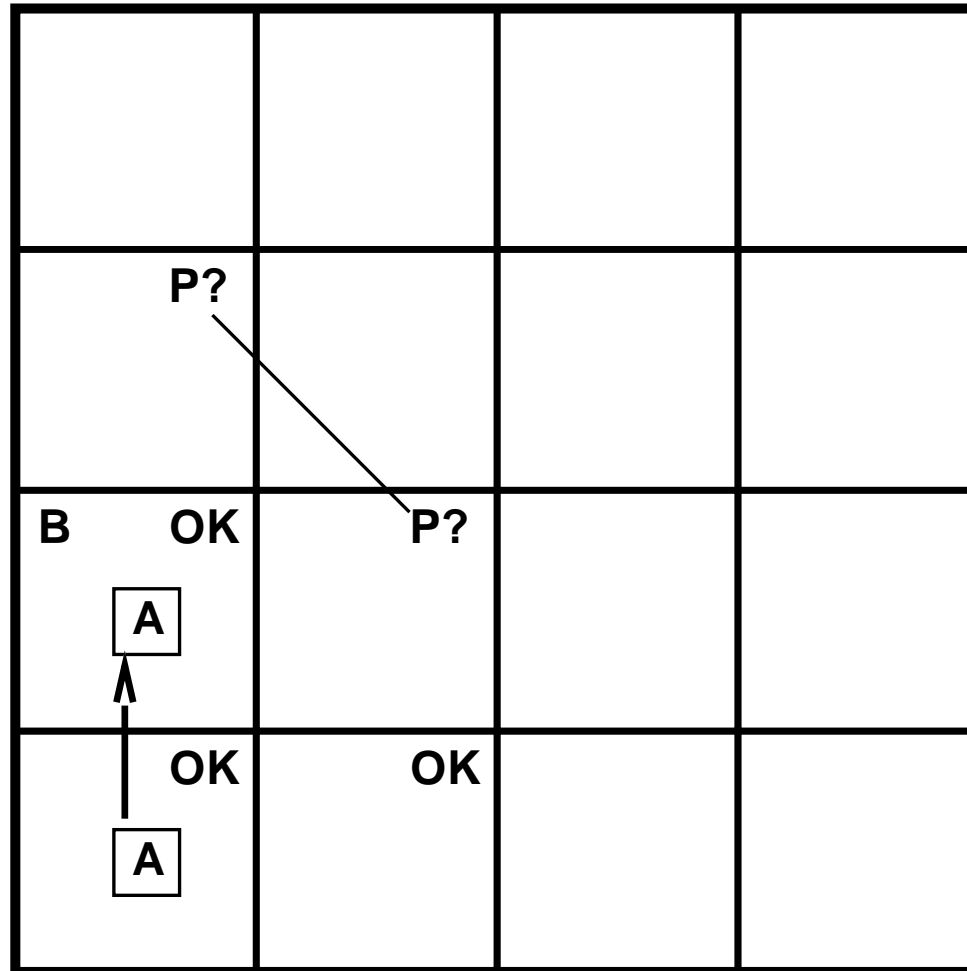
Exploring a wumpus world

OK			
OK A	OK		

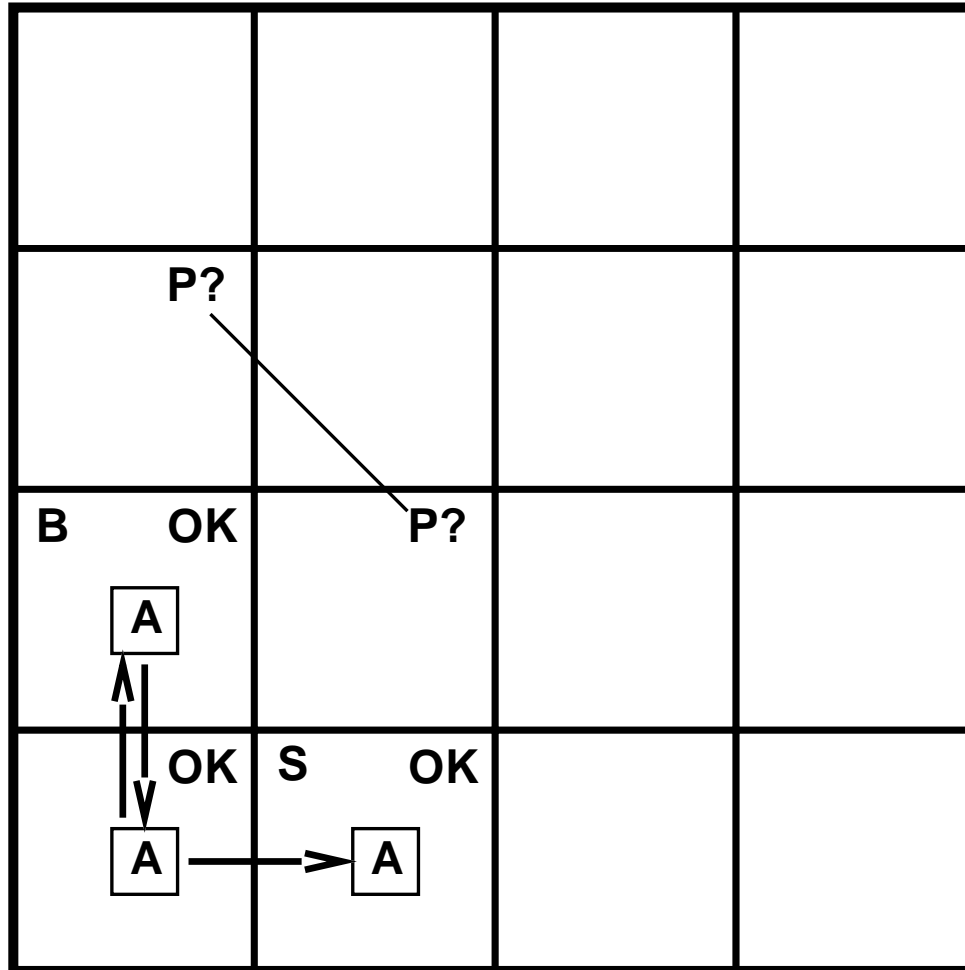
Exploring a wumpus world



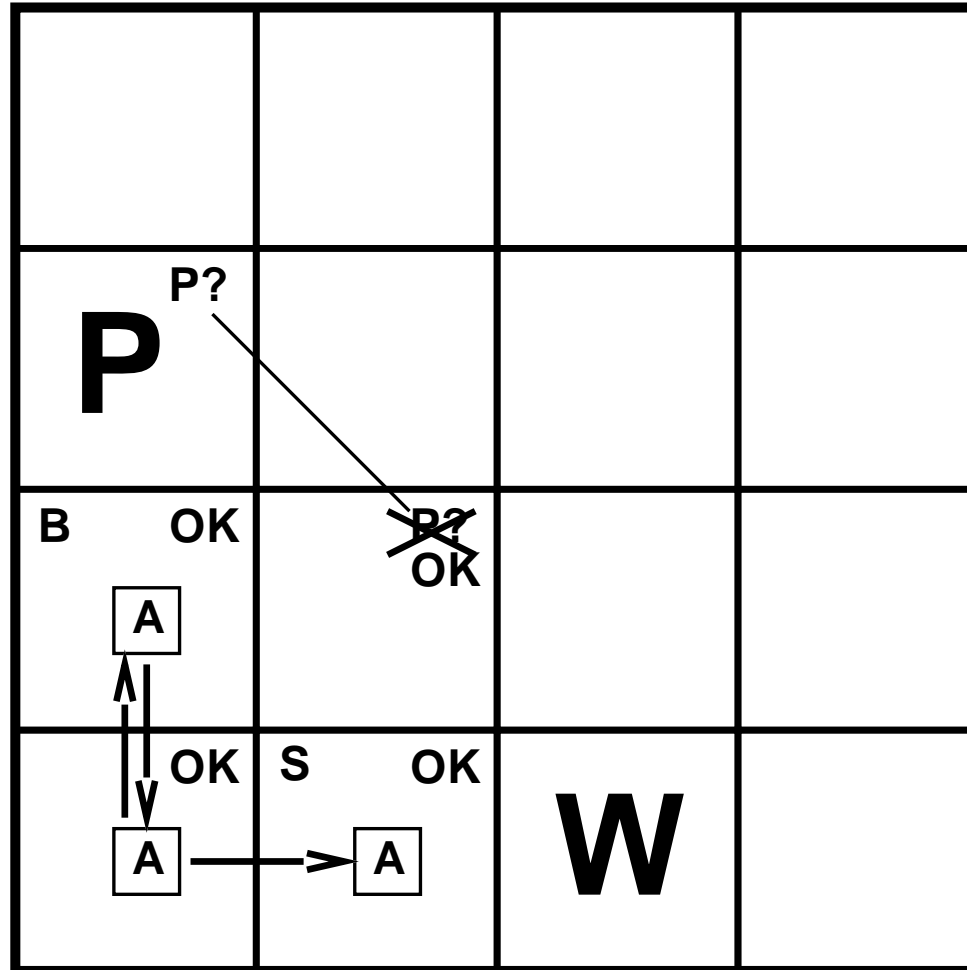
Exploring a wumpus world



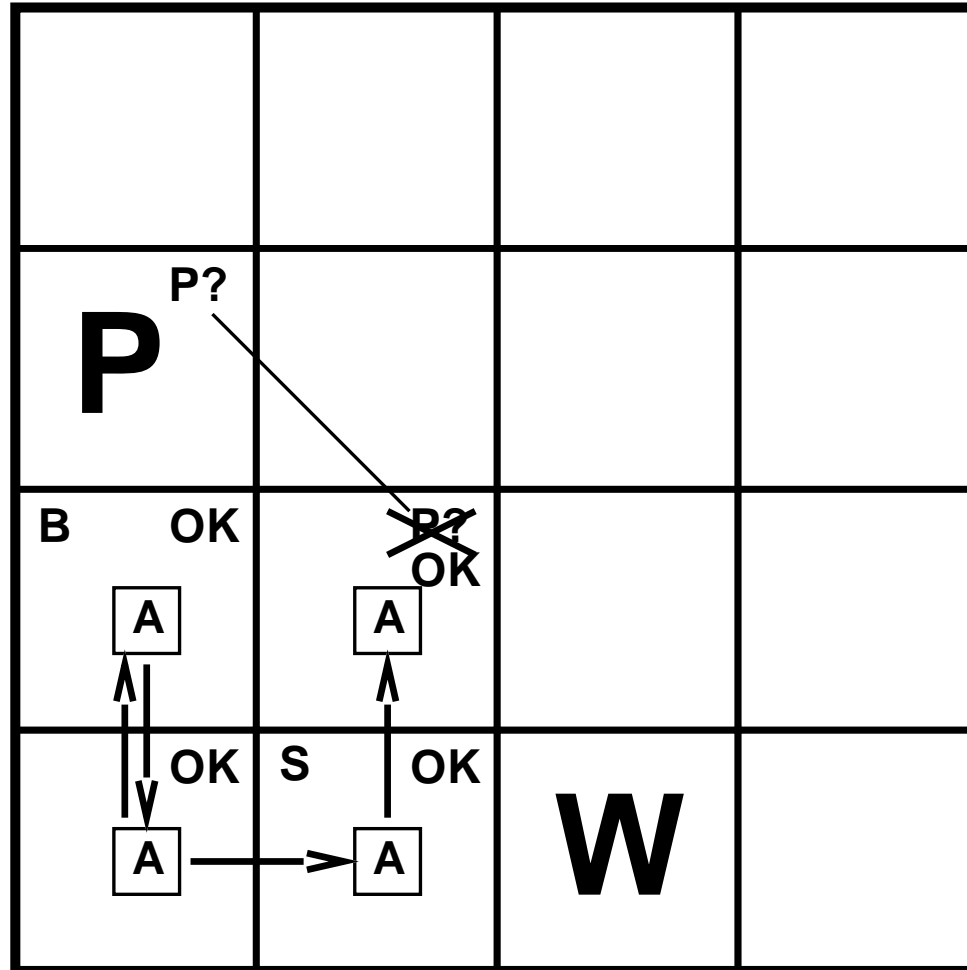
Exploring a wumpus world



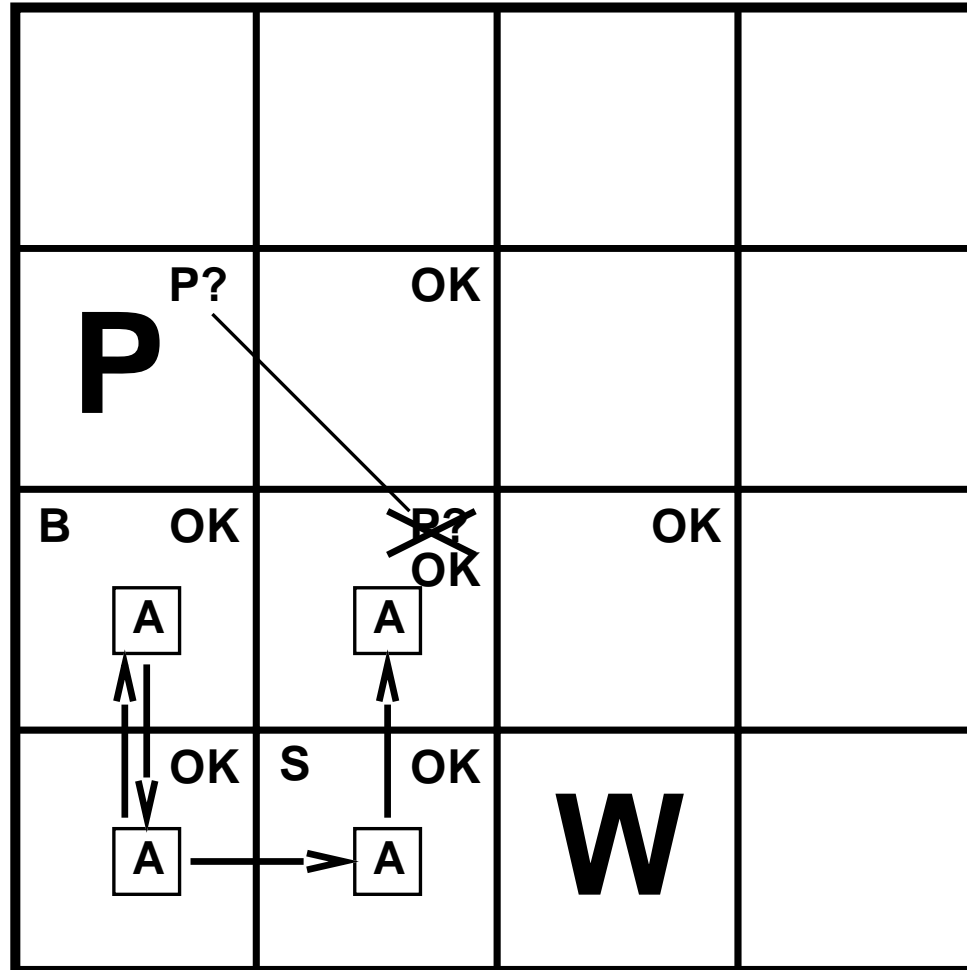
Exploring a wumpus world



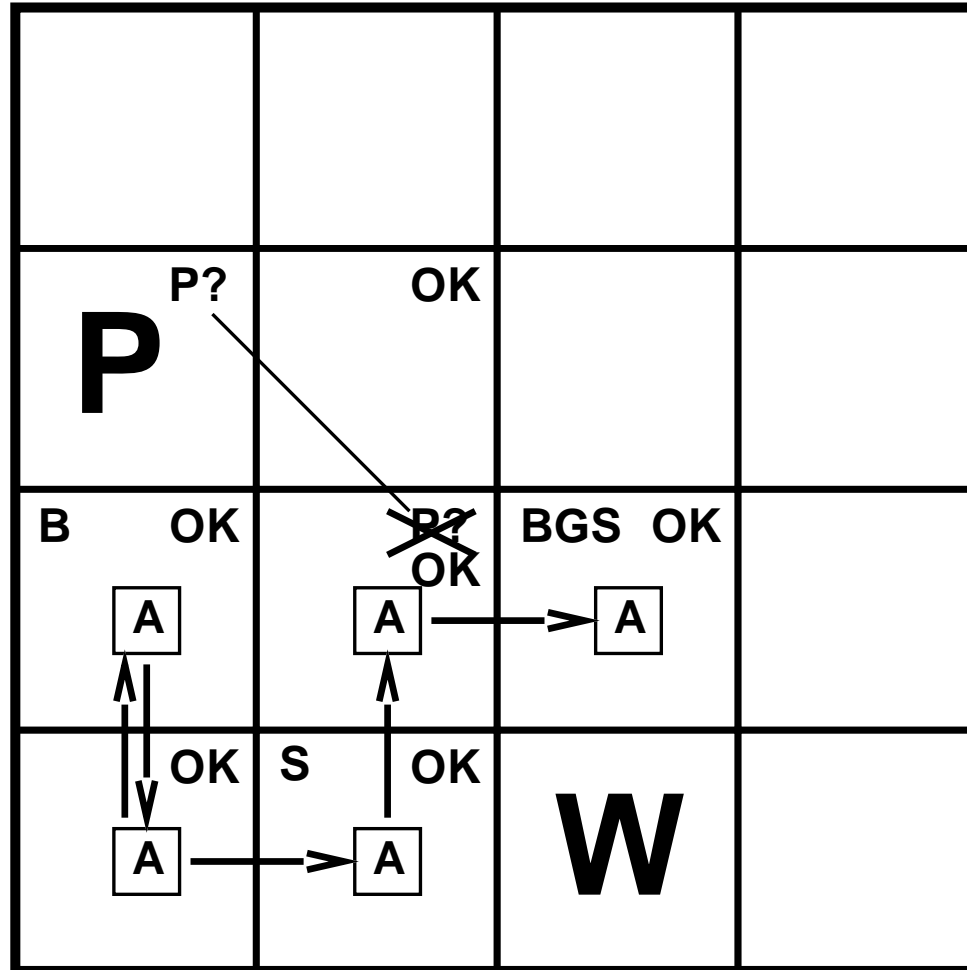
Exploring a wumpus world



Exploring a wumpus world



Exploring a wumpus world



Logic in general

Logics are formal languages for representing information such that conclusions can be drawn

Syntax defines the sentences in the language

Semantics define the “meaning” of sentences; i.e., define **truth** of a sentence in a world

E.g., the language of arithmetic

$x + 2 \geq y$ is a sentence; $x^2 + y >$ is not a sentence

$x + 2 \geq y$ is true iff the number $x + 2$ is no less than the number y

$x + 2 \geq y$ is true in a world where $x = 7, y = 1$

$x + 2 \geq y$ is false in a world where $x = 0, y = 6$

Entailment

Entailment means that one thing **follows from** another:

$$KB \models \alpha$$

Knowledge base KB entails sentence α
if and only if
 α is true in all worlds where KB is true

E.g., the KB containing “the Chiefs won” and “the Blues won”
entails “the Chiefs won”

E.g., $x + y = 4$ entails $4 = x + y$

Entailment is a relationship between sentences (i.e., **syntax**)
that is based on **semantics**

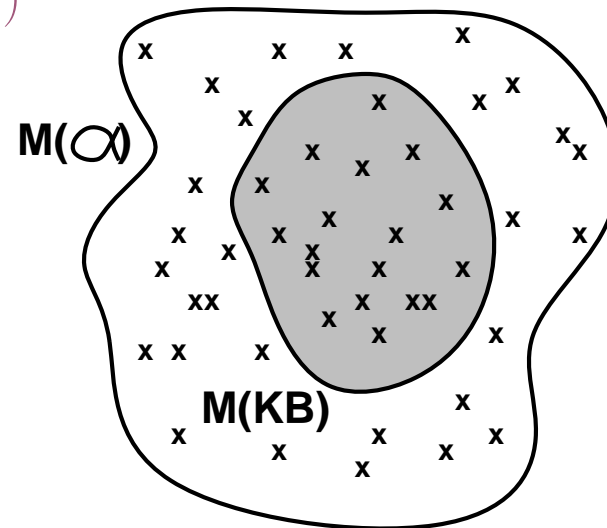
Models

We say m is a model of a sentence α if α is true in m

$M(\alpha)$ is the set of all models of α

Then $KB \models \alpha$ if and only if $M(KB) \subseteq M(\alpha)$

E.g. $KB =$ Chiefs won and Blues won
 $\alpha =$ Chiefs won

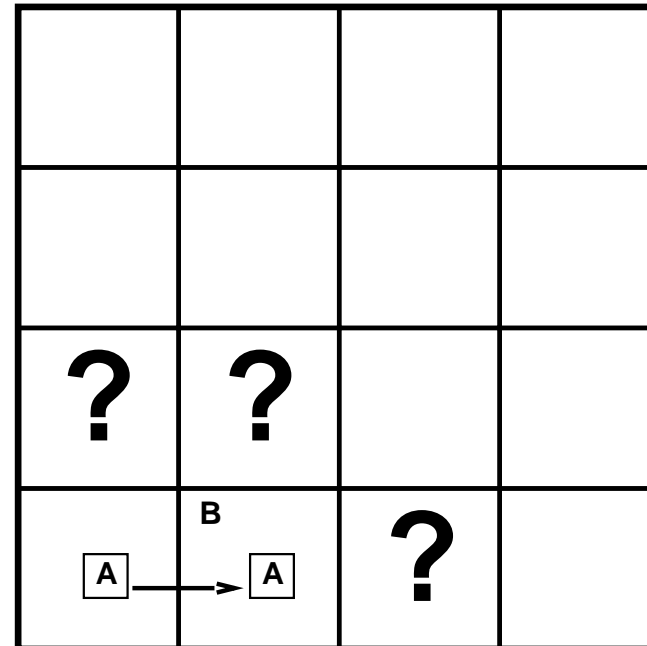


Entailment in the wumpus world

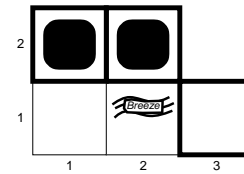
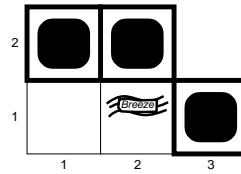
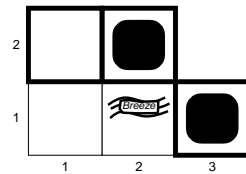
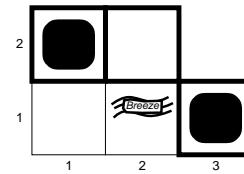
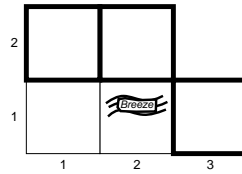
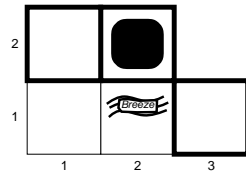
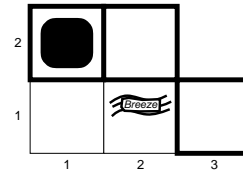
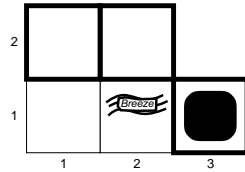
Situation after detecting nothing in [1,1],
moving right, breeze in [2,1]

Consider possible models for ?s
assuming only pits

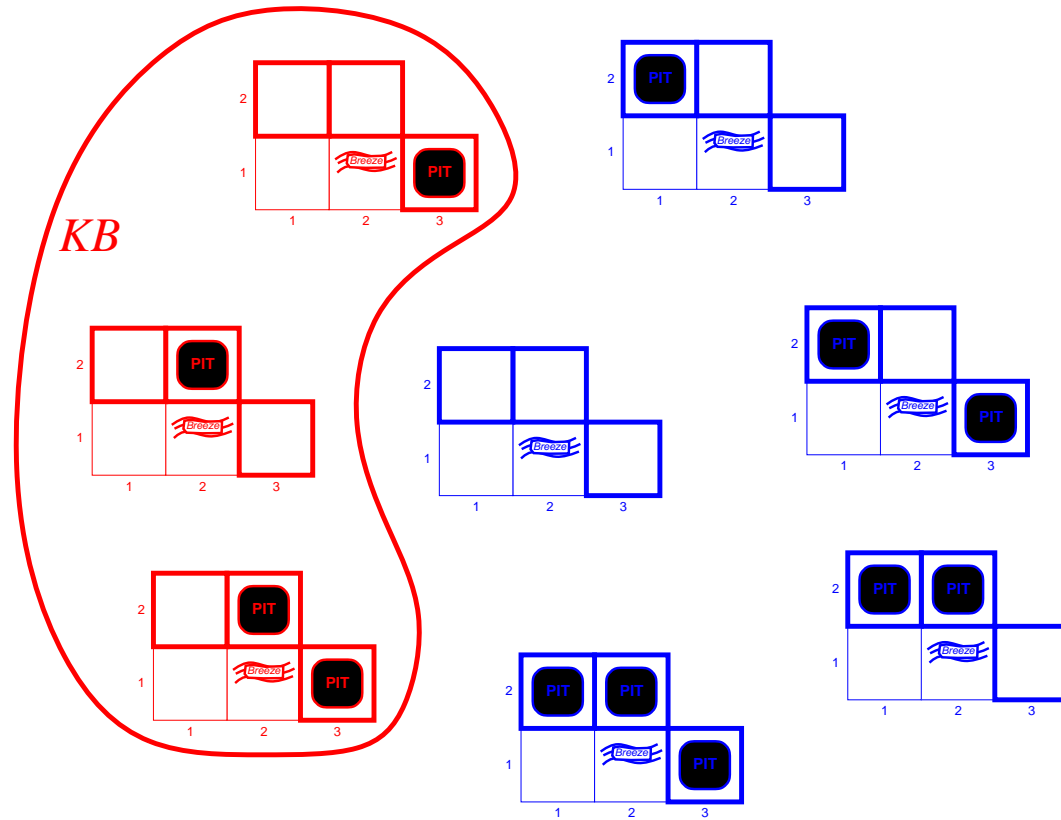
3 Boolean choices \Rightarrow 8 possible models



Wumpus models

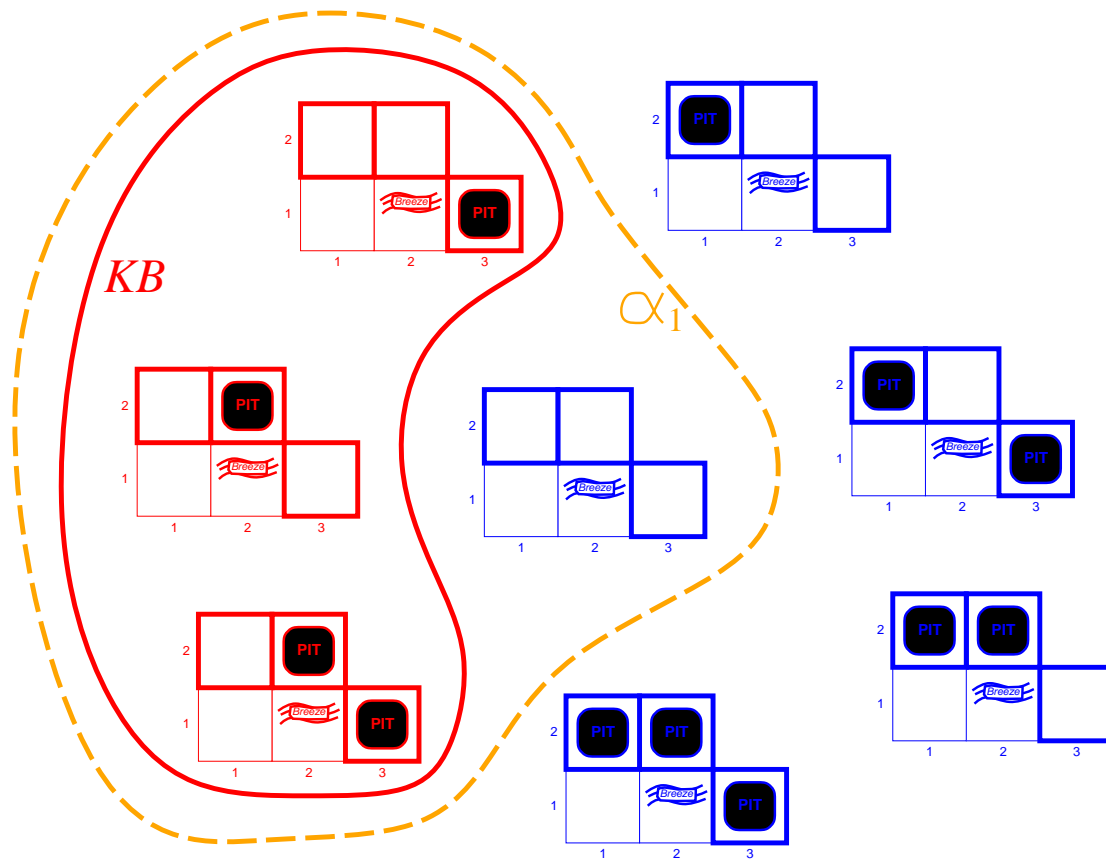


Wumpus models



KB = wumpus-world rules + observations

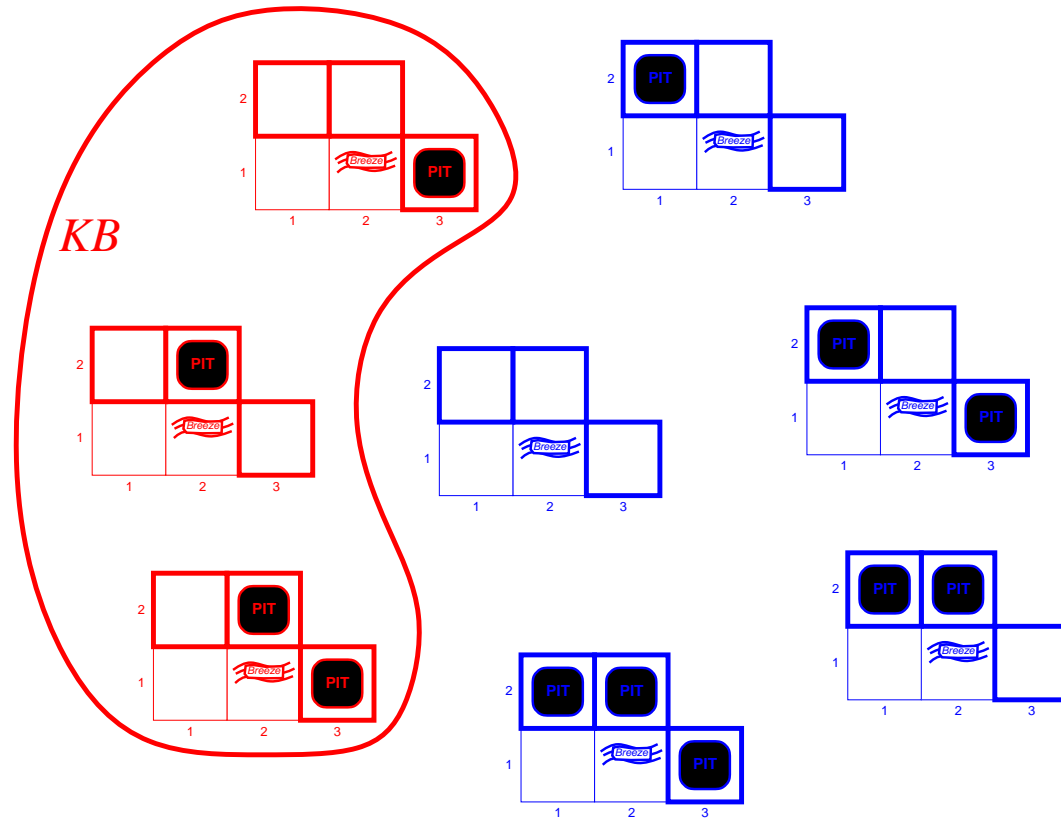
Wumpus models



KB = wumpus-world rules + observations

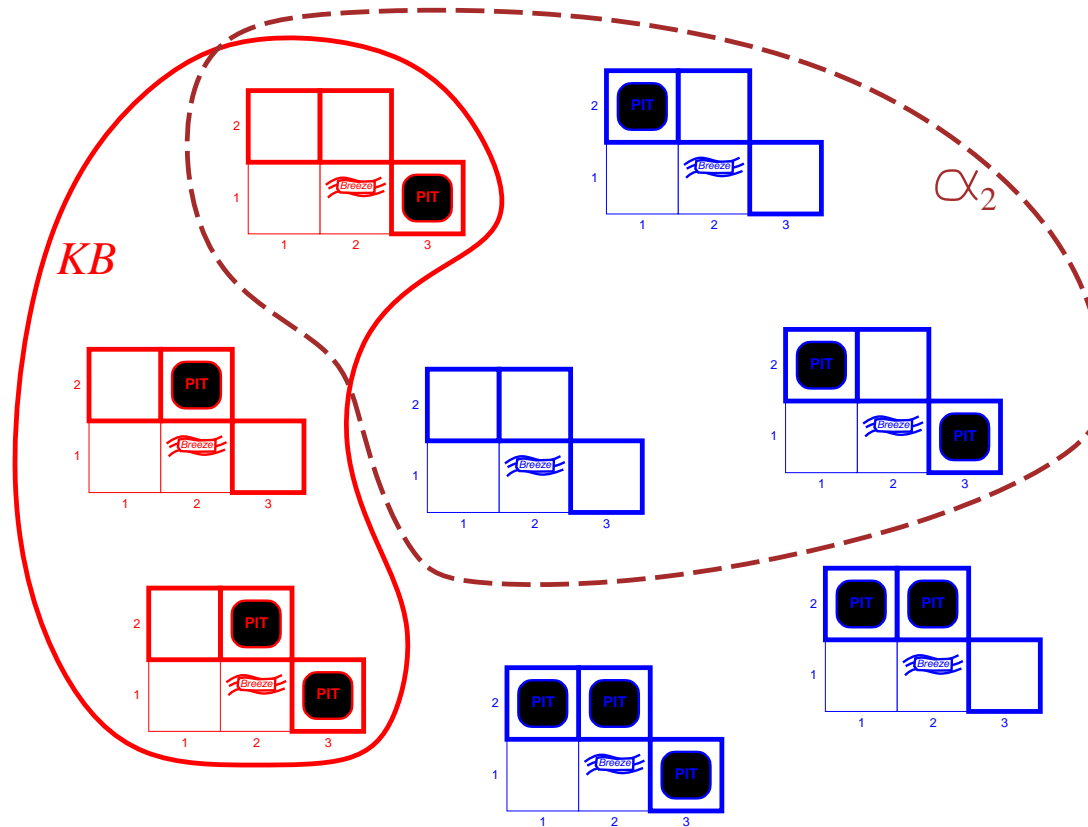
α_1 = “[1,2] is safe”, $KB \models \alpha_1$, proved by model checking

Wumpus models



$KB = \text{wumpus-world rules} + \text{observations}$

Wumpus models



KB = wumpus-world rules + observations

α_2 = "[2,2] is safe", $KB \not\models \alpha_2$

Inference

$KB \vdash_i \alpha$ = sentence α can be derived from KB by procedure i

Soundness: i is sound if

whenever $KB \vdash_i \alpha$, it is also true that $KB \models \alpha$

Completeness: i is complete if

whenever $KB \models \alpha$, it is also true that $KB \vdash_i \alpha$

Propositional logic: Syntax

Propositional logic is the simplest logic—illustrates basic ideas

The proposition symbols P_1, P_2 etc are sentences

If S is a sentence, $\neg S$ is a sentence (negation)

If S_1 and S_2 are sentences, $S_1 \wedge S_2$ is a sentence (conjunction)

If S_1 and S_2 are sentences, $S_1 \vee S_2$ is a sentence (disjunction)

If S_1 and S_2 are sentences, $S_1 \Rightarrow S_2$ is a sentence (implication)

If S_1 and S_2 are sentences, $S_1 \Leftrightarrow S_2$ is a sentence (biconditional)

Propositional logic: Semantics

Each model specifies true/false for each proposition symbol

E.g. $P_{1,2}$ $P_{2,2}$ $P_{3,1}$
true true false

(With these symbols, 8 possible models, can be enumerated automatically.)

Rules for evaluating truth with respect to a model m :

$\neg S$ is true iff	S	is false
$S_1 \wedge S_2$ is true iff	S_1	is true and S_2 is true
$S_1 \vee S_2$ is true iff	S_1	is true or S_2 is true
$S_1 \Rightarrow S_2$ is true iff	S_1	is false or S_2 is true
i.e., is false iff	S_1	is true and S_2 is false
$S_1 \Leftrightarrow S_2$ is true iff	$S_1 \Rightarrow S_2$ is true	and $S_2 \Rightarrow S_1$ is true

Simple recursive process evaluates an arbitrary sentence, e.g.,

$\neg P_{1,2} \wedge (P_{2,2} \vee P_{3,1}) = \textit{true} \wedge (\textit{false} \vee \textit{true}) = \textit{true} \wedge \textit{true} = \textit{true}$

Truth tables for connectives

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
<i>false</i>	<i>false</i>	<i>true</i>	<i>false</i>	<i>false</i>	<i>true</i>	<i>true</i>
<i>false</i>	<i>true</i>	<i>true</i>	<i>false</i>	<i>true</i>	<i>true</i>	<i>false</i>
<i>true</i>	<i>false</i>	<i>false</i>	<i>false</i>	<i>true</i>	<i>false</i>	<i>false</i>
<i>true</i>	<i>true</i>	<i>false</i>	<i>true</i>	<i>true</i>	<i>true</i>	<i>true</i>

Wumpus world sentences

Let $P_{i,j}$ be true if there is a pit in $[i, j]$.

Let $B_{i,j}$ be true if there is a breeze in $[i, j]$.

There was no pit in $[1, 1]$ in the example discussed earlier:

$$R_1 : \neg P_{1,1}$$

“A square is breezy **if and only if** it is adjacent to a pit.”

$$R_2 : B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$$

$$R_3 : B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$$

There is no breeze in $[1, 1]$, but there is one in $[2, 1]$:

$$R_4 : \neg B_{1,1}$$

$$R_5 : B_{2,1}$$

Truth tables for inference

$B_{1,1}$	$B_{2,1}$	$P_{1,1}$	$P_{1,2}$	$P_{2,1}$	$P_{2,2}$	$P_{3,1}$	R_1	R_2	R_3	R_4	R_5	KB
<i>false</i>	<i>false</i>	<i>false</i>	<i>false</i>	<i>false</i>	<i>false</i>	<i>false</i>	<i>true</i>	<i>true</i>	<i>true</i>	<i>true</i>	<i>false</i>	<i>false</i>
<i>false</i>	<i>false</i>	<i>false</i>	<i>false</i>	<i>false</i>	<i>false</i>	<i>true</i>	<i>true</i>	<i>true</i>	<i>false</i>	<i>true</i>	<i>false</i>	<i>false</i>
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
<i>false</i>	<i>true</i>	<i>false</i>	<i>false</i>	<i>false</i>	<i>false</i>	<i>false</i>	<i>true</i>	<i>true</i>	<i>false</i>	<i>true</i>	<i>true</i>	<i>false</i>
<i>false</i>	<i>true</i>	<i>false</i>	<i>false</i>	<i>false</i>	<i>false</i>	<i>true</i>	<i>true</i>	<i>true</i>	<i>true</i>	<i>true</i>	<i>true</i>	<u><i>true</i></u>
<i>false</i>	<i>true</i>	<i>false</i>	<i>false</i>	<i>false</i>	<i>true</i>	<i>false</i>	<i>true</i>	<i>true</i>	<i>true</i>	<i>true</i>	<i>true</i>	<u><i>true</i></u>
<i>false</i>	<i>true</i>	<i>false</i>	<i>false</i>	<i>false</i>	<i>true</i>	<i>true</i>	<i>true</i>	<i>true</i>	<i>true</i>	<i>true</i>	<i>true</i>	<u><i>true</i></u>
<i>false</i>	<i>true</i>	<i>false</i>	<i>false</i>	<i>true</i>	<i>false</i>	<i>false</i>	<i>true</i>	<i>false</i>	<i>false</i>	<i>true</i>	<i>true</i>	<i>false</i>
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
<i>true</i>	<i>true</i>	<i>true</i>	<i>true</i>	<i>true</i>	<i>true</i>	<i>true</i>	<i>false</i>	<i>true</i>	<i>true</i>	<i>false</i>	<i>true</i>	<i>false</i>

Enumerate rows (different assignments to symbols),
 if **KB** is true in row, check that α is too

Inference by enumeration

Depth-first enumeration of all models is sound and complete

```
function TT-ENTAILS?(KB,  $\alpha$ ) returns true or false
  inputs: KB, the knowledge base, a sentence in propositional logic
            $\alpha$ , the query, a sentence in propositional logic
  symbols  $\leftarrow$  a list of the proposition symbols in KB and  $\alpha$ 
  return TT-CHECK-ALL(KB,  $\alpha$ , symbols, [])
```

```
function TT-CHECK-ALL(KB,  $\alpha$ , symbols, model) returns true or false
  if EMPTY?(symbols) then
    if PL-TRUE?(KB, model) then return PL-TRUE?( $\alpha$ , model)
    else return true
  else do
    P  $\leftarrow$  FIRST(symbols); rest  $\leftarrow$  REST(symbols)
    return TT-CHECK-ALL(KB,  $\alpha$ , rest, EXTEND(P, true, model)) and
           TT-CHECK-ALL(KB,  $\alpha$ , rest, EXTEND(P, false, model))
```

$O(2^n)$ for n symbols; problem is **co-NP-complete**

Logical equivalence

Two sentences are **logically equivalent** iff true in same models:

$\alpha \equiv \beta$ if and only if $\alpha \models \beta$ and $\beta \models \alpha$

$$(\alpha \wedge \beta) \equiv (\beta \wedge \alpha) \quad \text{commutativity of } \wedge$$

$$(\alpha \vee \beta) \equiv (\beta \vee \alpha) \quad \text{commutativity of } \vee$$

$$((\alpha \wedge \beta) \wedge \gamma) \equiv (\alpha \wedge (\beta \wedge \gamma)) \quad \text{associativity of } \wedge$$

$$((\alpha \vee \beta) \vee \gamma) \equiv (\alpha \vee (\beta \vee \gamma)) \quad \text{associativity of } \vee$$

$$\neg(\neg\alpha) \equiv \alpha \quad \text{double-negation elimination}$$

$$(\alpha \Rightarrow \beta) \equiv (\neg\beta \Rightarrow \neg\alpha) \quad \text{contraposition}$$

$$(\alpha \Rightarrow \beta) \equiv (\neg\alpha \vee \beta) \quad \text{implication elimination}$$

$$(\alpha \Leftrightarrow \beta) \equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)) \quad \text{biconditional elimination}$$

$$\neg(\alpha \wedge \beta) \equiv (\neg\alpha \vee \neg\beta) \quad \text{De Morgan}$$

$$\neg(\alpha \vee \beta) \equiv (\neg\alpha \wedge \neg\beta) \quad \text{De Morgan}$$

$$(\alpha \wedge (\beta \vee \gamma)) \equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)) \quad \text{distributivity of } \wedge \text{ over } \vee$$

$$(\alpha \vee (\beta \wedge \gamma)) \equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \quad \text{distributivity of } \vee \text{ over } \wedge$$

Validity and satisfiability

A sentence is **valid** if it is true in **all** models,

e.g., *True*, $A \vee \neg A$, $A \Rightarrow A$, $(A \wedge (A \Rightarrow B)) \Rightarrow B$

Validity is connected to entailment via the following:

$KB \models \alpha$ if and only if $(KB \Rightarrow \alpha)$ is valid

A sentence is **satisfiable** if it is true in **some** model

e.g., $A \vee B$, C

A sentence is **unsatisfiable** if it is true in **no** models

e.g., $A \wedge \neg A$

Satisfiability is connected to entailment via the following:

$KB \models \alpha$ if and only if $(KB \wedge \neg \alpha)$ is unsatisfiable

i.e., prove α by *reductio ad absurdum*

Proof methods

Proof methods divide into (roughly) two kinds:

Application of inference rules

- Legitimate (sound) generation of new sentences from old
- **Proof** = a sequence of inference rule applications
 - Can use inference rules as operators in a standard search alg.
- Typically require translation of sentences into a **normal form**

Model checking

- truth table enumeration (always exponential in n)
- improved backtracking, e.g., Davis–Putnam–Logemann–Loveland
- heuristic search in model space (sound but incomplete)
 - e.g., min-conflicts-like hill-climbing algorithms

Forward and backward chaining

Horn Form (restricted)

KB = **conjunction** of **Horn clauses**

Horn clause =

◇ proposition symbol; or

◇ (conjunction of symbols) \Rightarrow symbol

E.g., $C \wedge (B \Rightarrow A) \wedge (C \wedge D \Rightarrow B)$

Modus Ponens (for Horn Form): complete for Horn KBs

$$\frac{\alpha_1, \dots, \alpha_n, \quad \alpha_1 \wedge \dots \wedge \alpha_n \Rightarrow \beta}{\beta}$$

Can be used with **forward chaining** or **backward chaining**.
These algorithms are very natural and run in **linear** time

Forward chaining

Idea: fire any rule whose premises are satisfied in the *KB*,
add its conclusion to the *KB*, until query is found

$$P \Rightarrow Q$$

$$L \wedge M \Rightarrow P$$

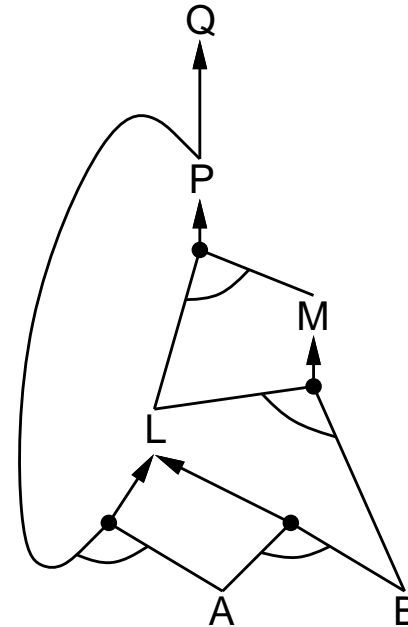
$$B \wedge L \Rightarrow M$$

$$A \wedge P \Rightarrow L$$

$$A \wedge B \Rightarrow L$$

A

B



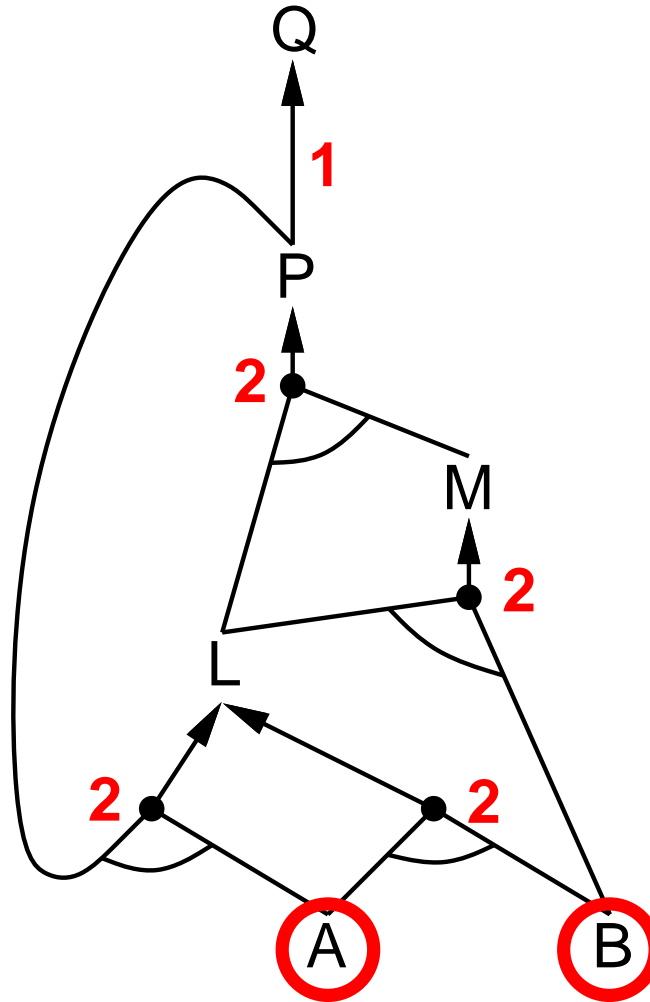
Forward chaining algorithm

```
function PL-FC-ENTAILS?(KB, q) returns true or false
  inputs: KB, the knowledge base, a set of propositional Horn clauses
           q, the query, a proposition symbol
  local variables: count, a table, indexed by clause, initially the number of premises
                    inferred, a table, indexed by symbol, each entry initially false
                    agenda, a list of symbols, initially the symbols known in KB

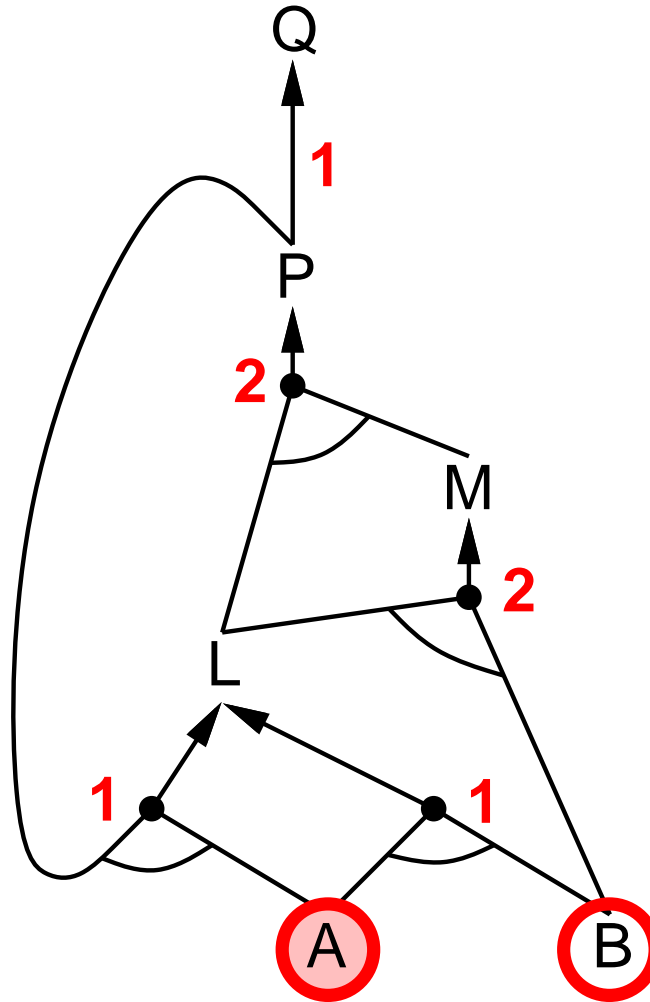
  while agenda is not empty do
    p ← POP(agenda)
    unless inferred[p] do
      inferred[p] ← true
      for each Horn clause c in whose premise p appears do
        decrement count[c]
        if count[c] = 0 then do
          if HEAD[c] = q then return true
          PUSH(HEAD[c], agenda)

  return false
```

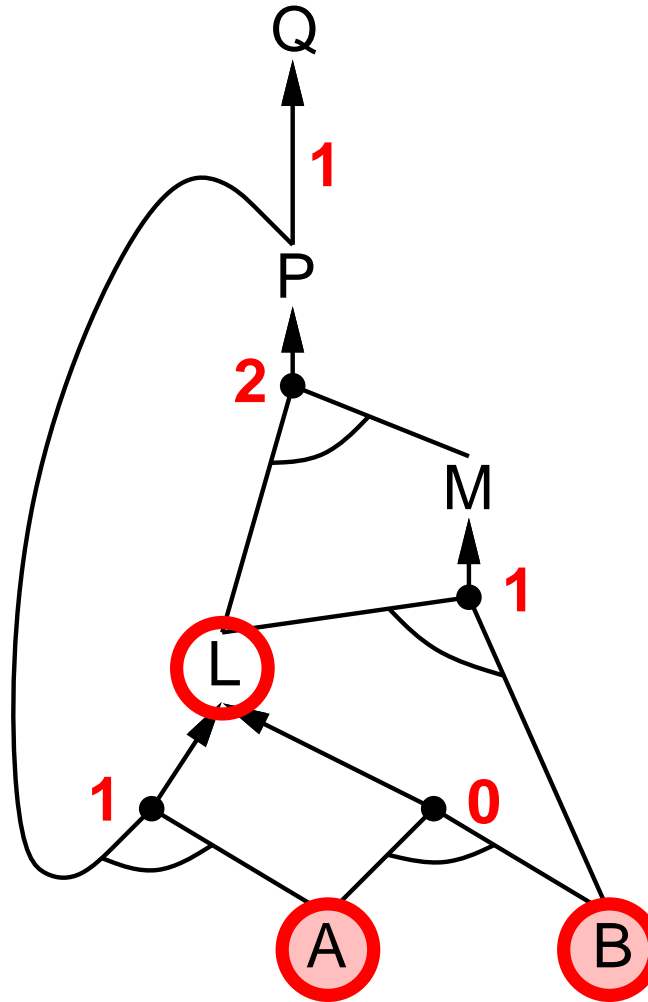
Forward chaining example



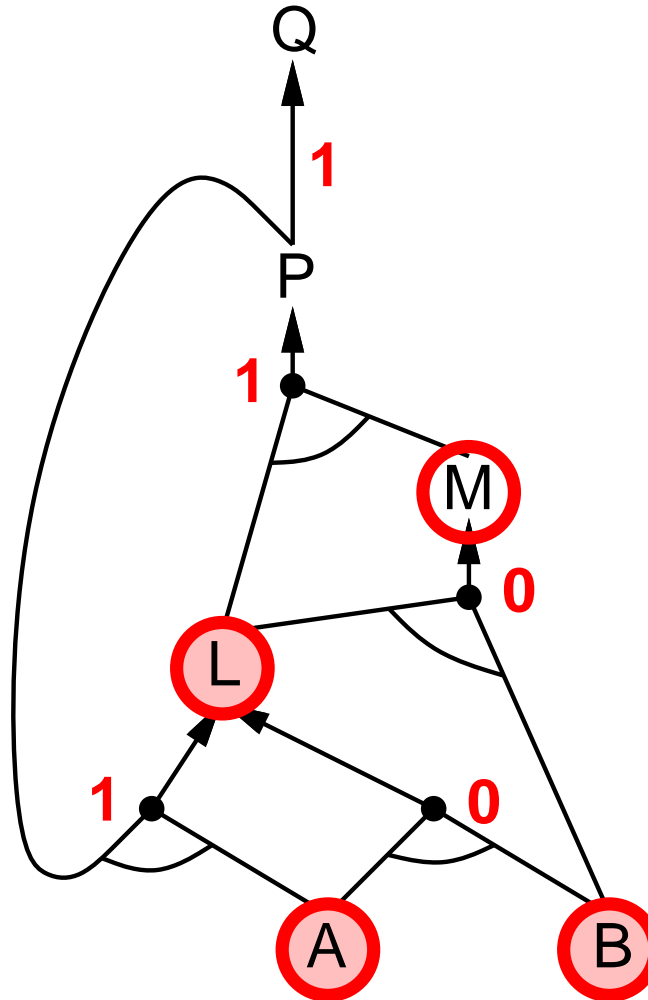
Forward chaining example



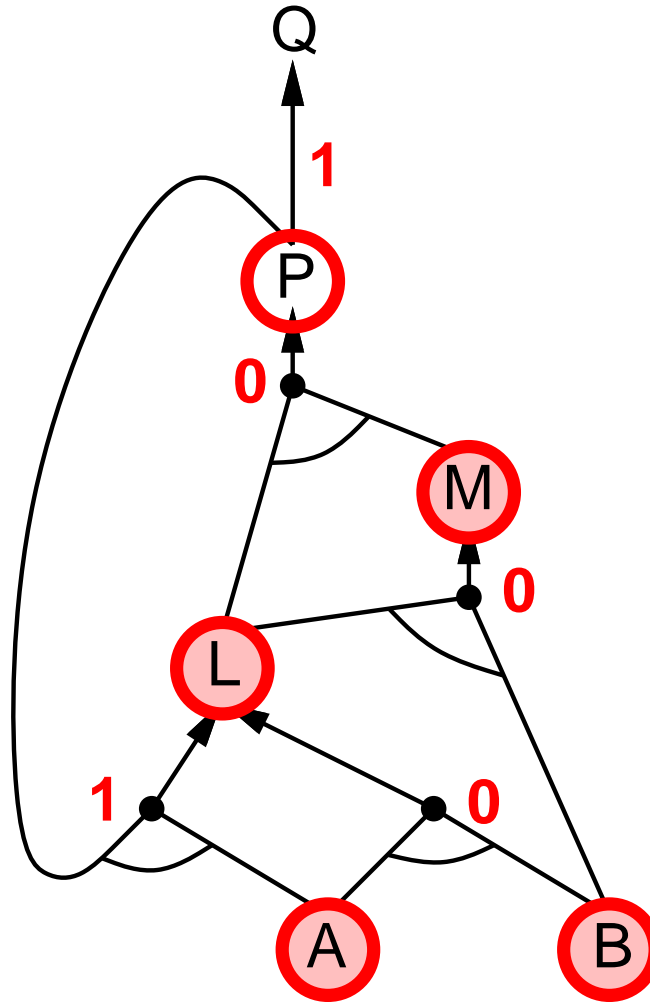
Forward chaining example



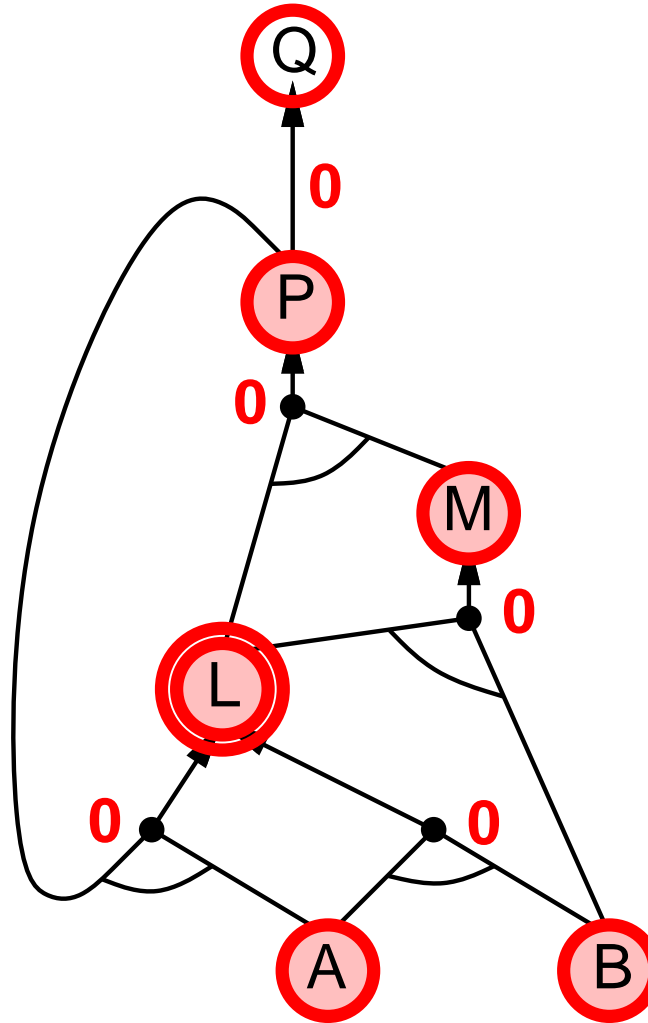
Forward chaining example



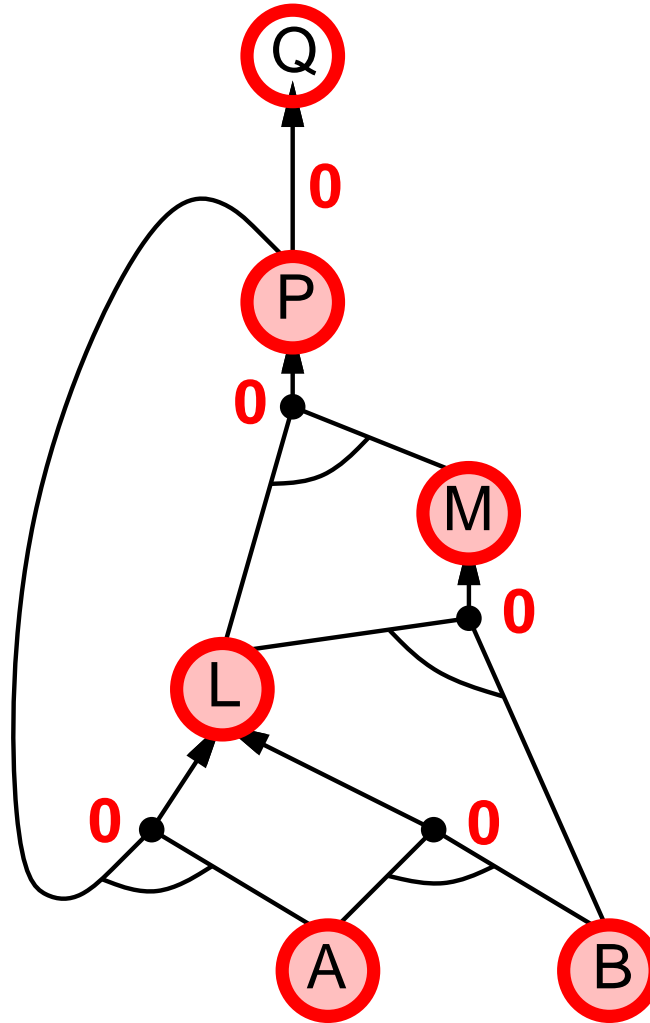
Forward chaining example



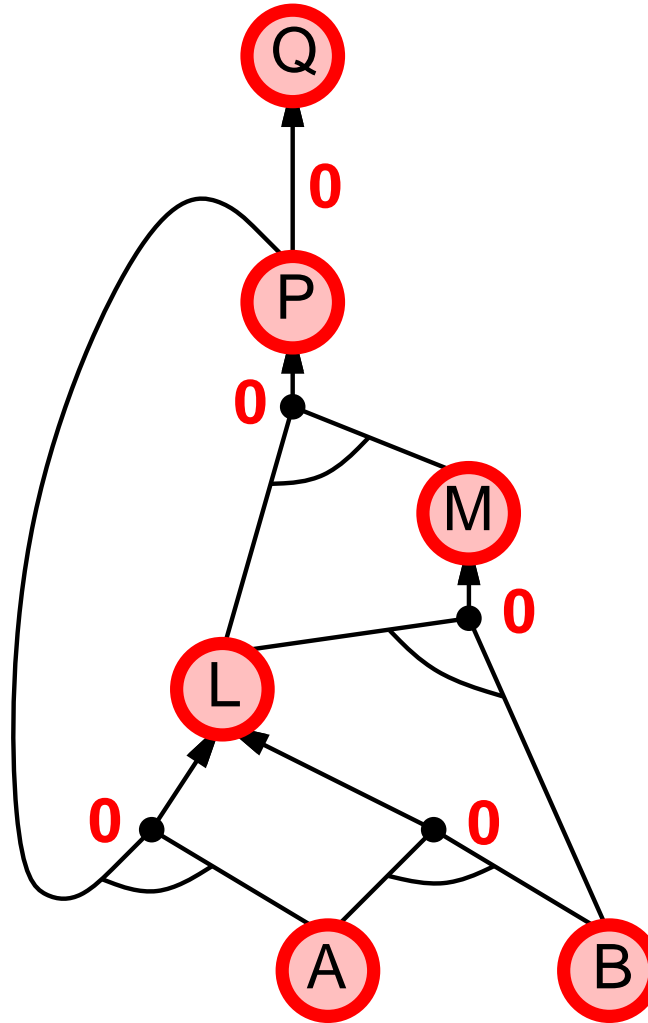
Forward chaining example



Forward chaining example



Forward chaining example



Backward chaining

Idea: work backwards from the query q :

to prove q by BC,

check if q is known already, or

prove by BC all premises of some rule concluding q

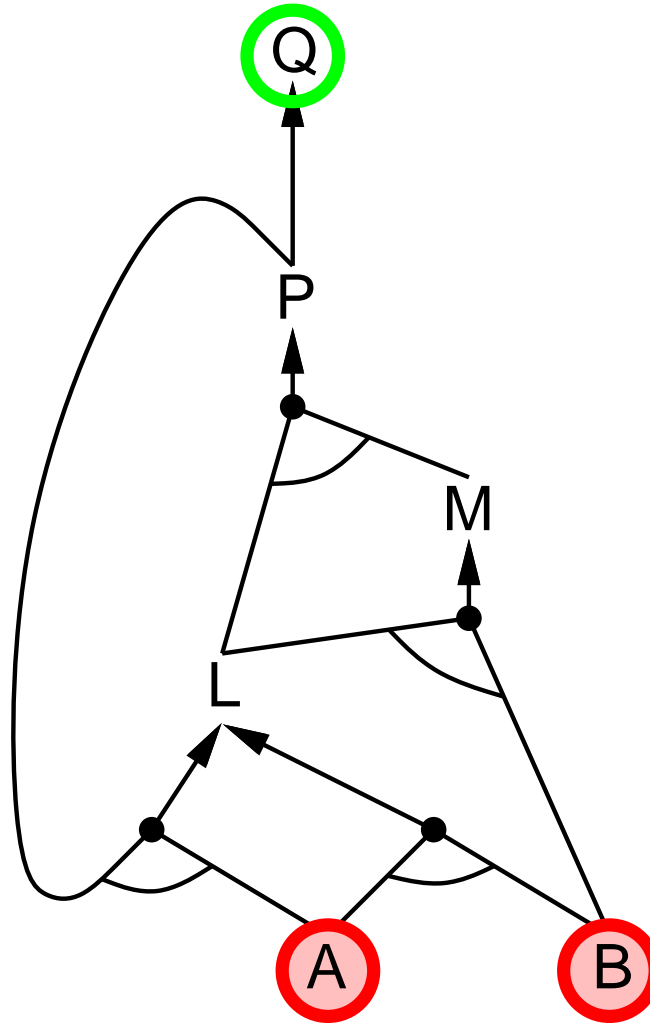
Avoid loops: check if new subgoal is already on the goal stack

Avoid repeated work: check if new subgoal

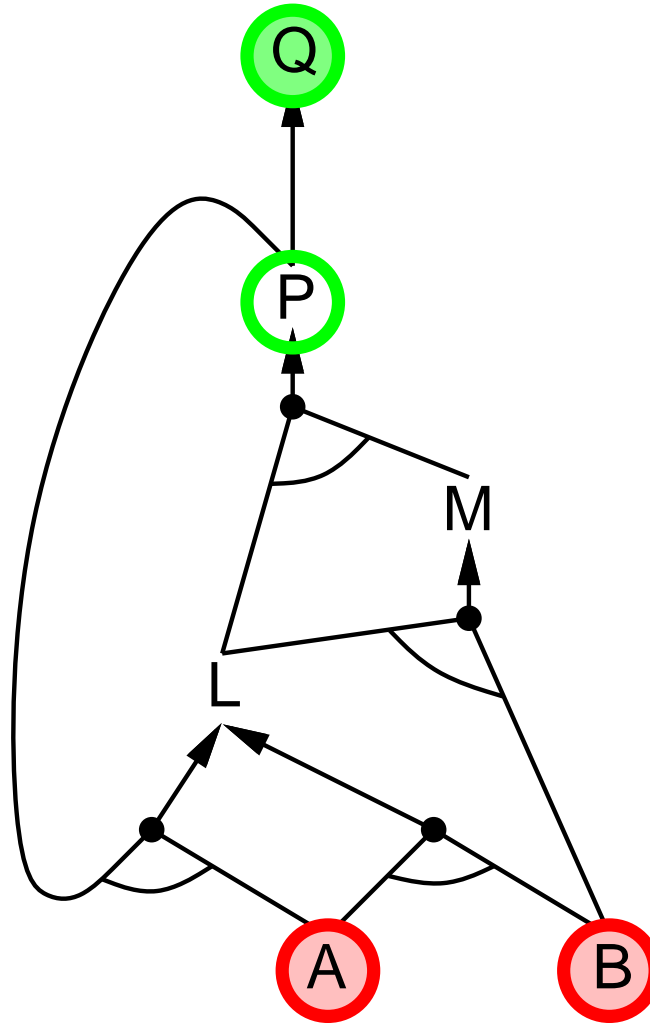
1) has already been proved true, or

2) has already failed

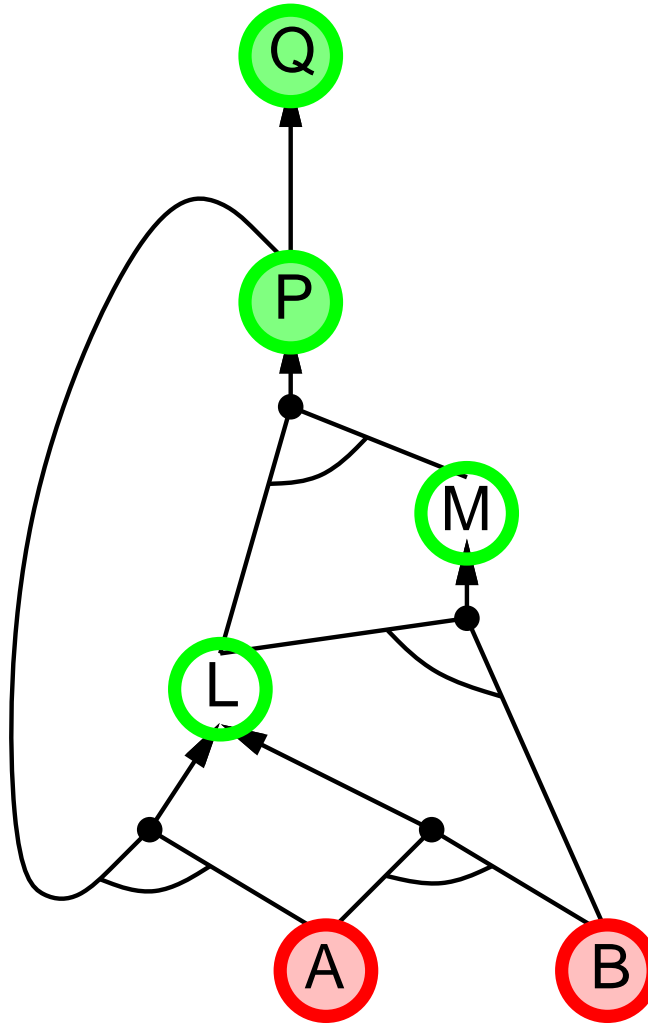
Backward chaining example



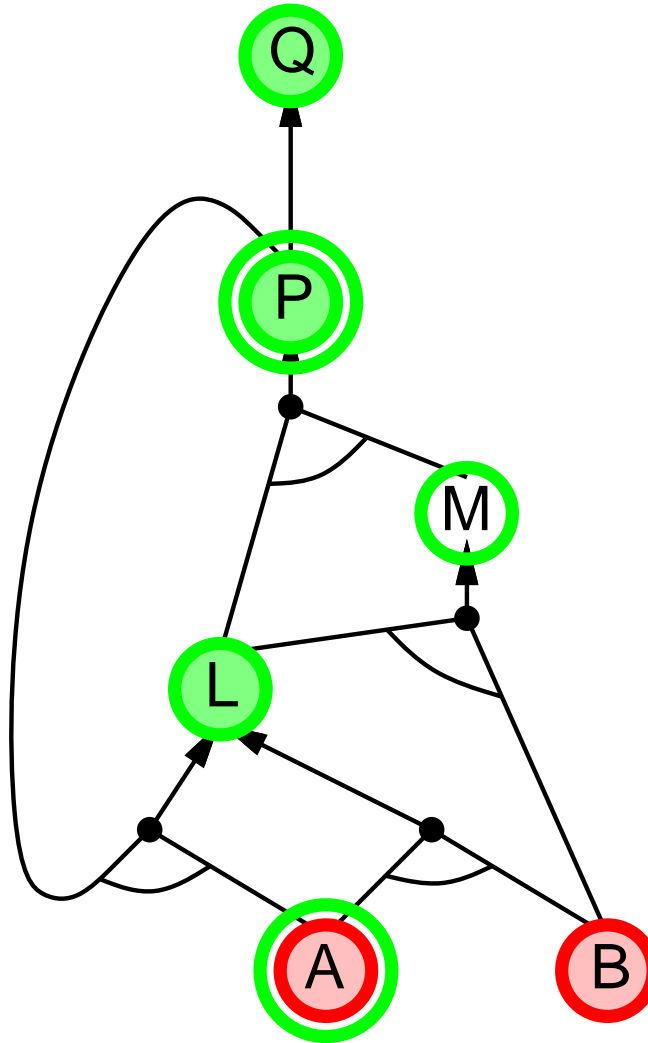
Backward chaining example



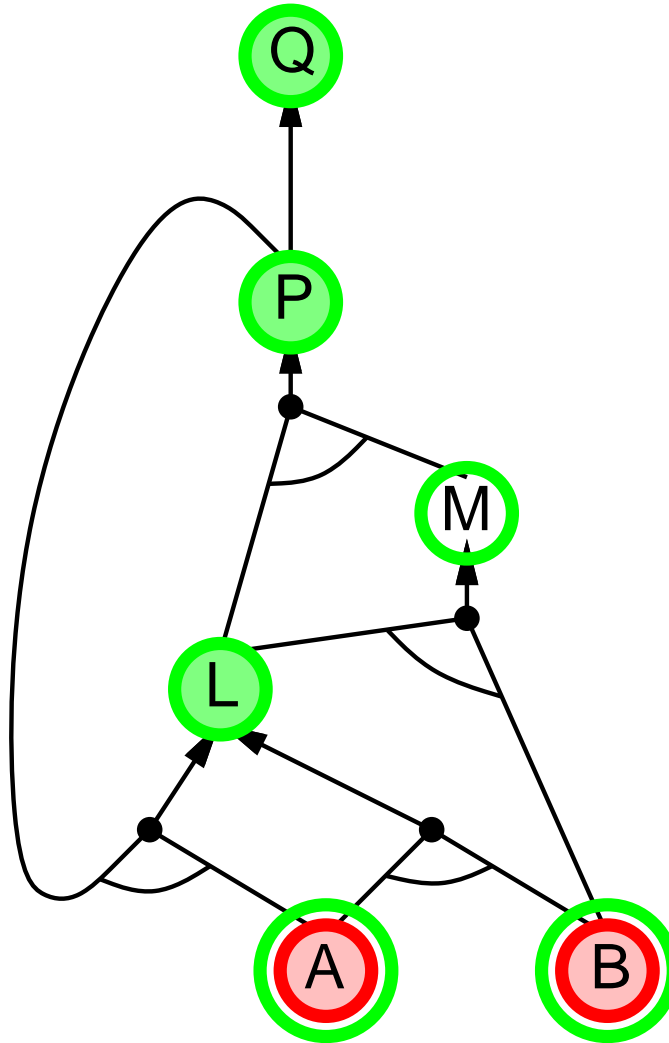
Backward chaining example



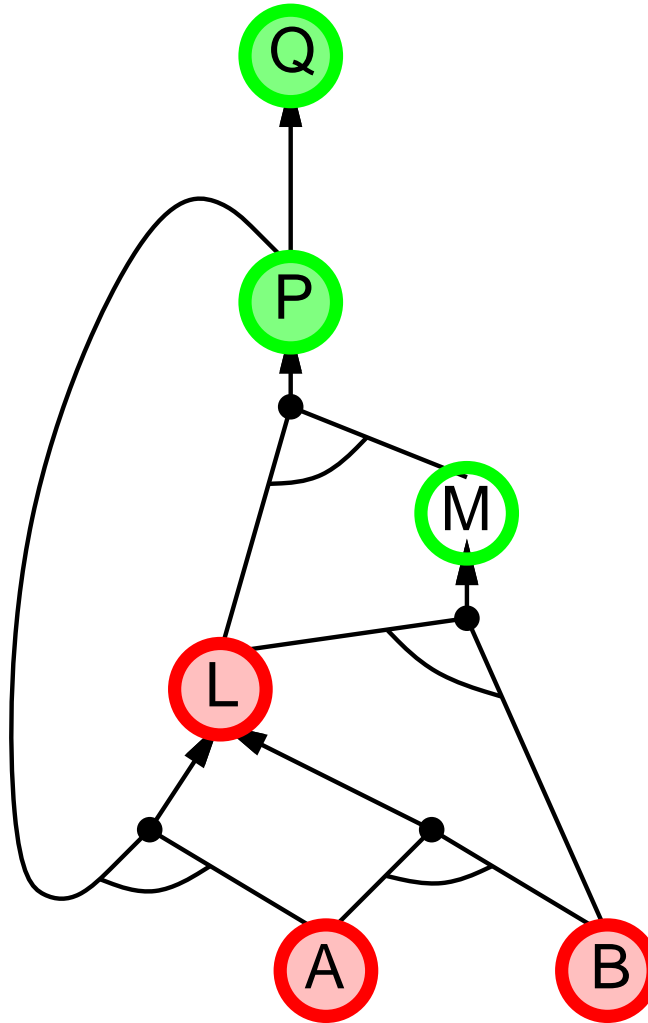
Backward chaining example



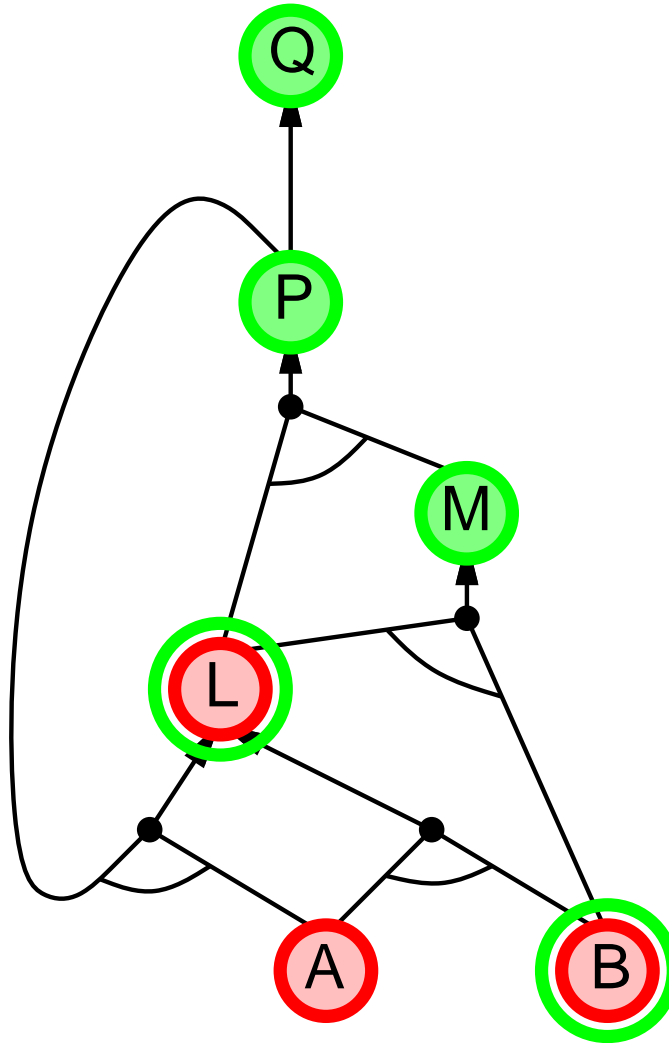
Backward chaining example



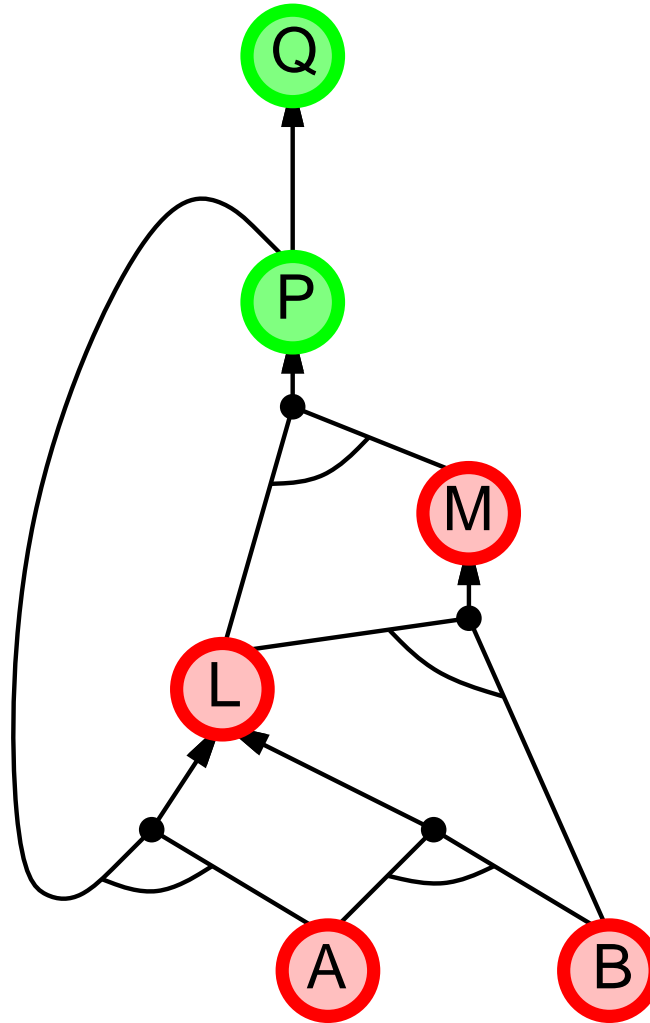
Backward chaining example



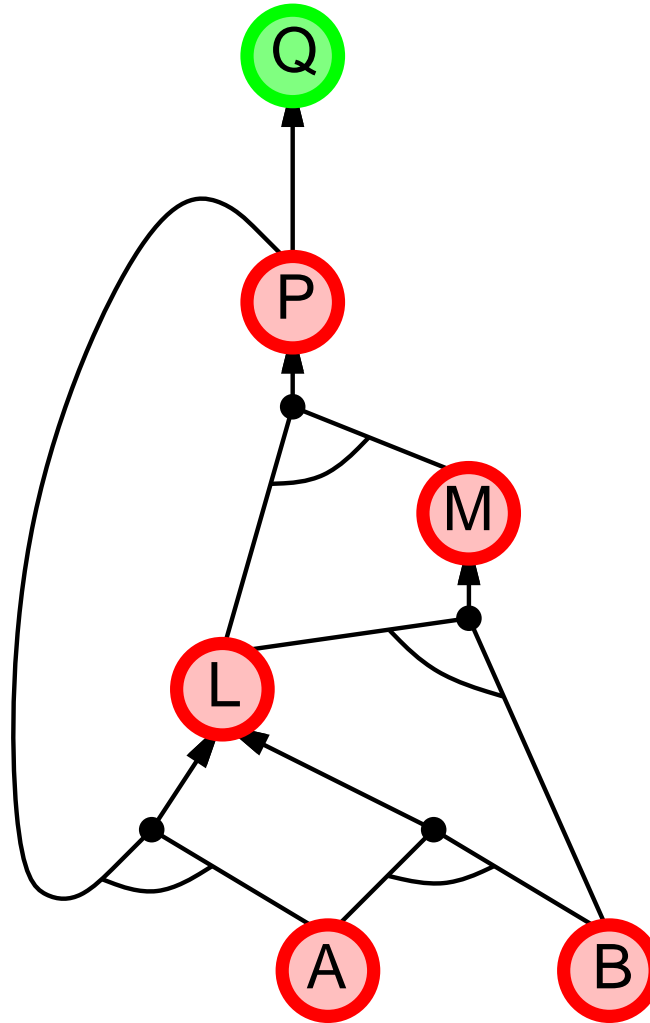
Backward chaining example



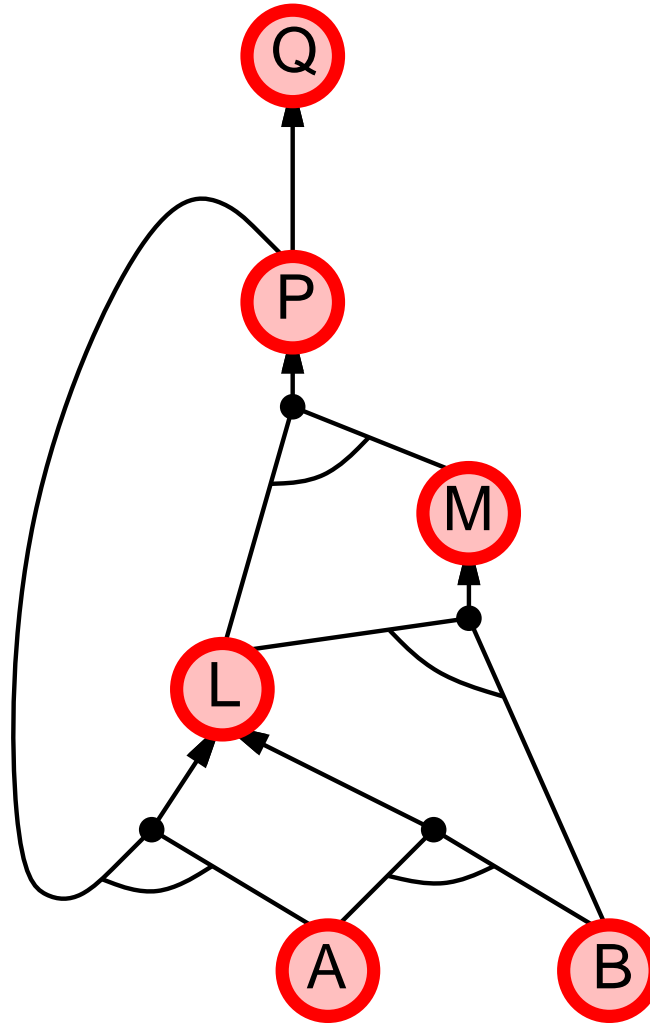
Backward chaining example



Backward chaining example



Backward chaining example



Forward vs. backward chaining

FC is data-driven

May do lots of work that is irrelevant to the goal

BC is goal-driven

Complexity of BC can be **much less** than linear in size of KB

Resolution

Conjunctive Normal Form (CNF—universal)
conjunction of **disjunctions** of **literals**
clauses

E.g., $(A \vee \neg B) \wedge (B \vee \neg C \vee \neg D)$

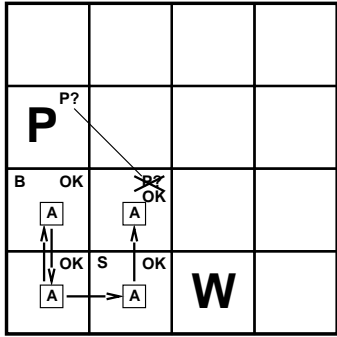
Resolution inference rule (for CNF): complete for propositional logic

$$\frac{l_1 \vee \dots \vee l_k, \quad m_1 \vee \dots \vee m_n}{l_1 \vee \dots \vee l_{i-1} \vee l_{i+1} \vee \dots \vee l_k \vee m_1 \vee \dots \vee m_{j-1} \vee m_{j+1} \vee \dots \vee m_n}$$

where l_i and m_j are complementary literals. E.g.,

$$\frac{P_{1,3} \vee P_{2,2}, \quad \neg P_{2,2}}{P_{1,3}}$$

Resolution is sound and complete for propositional logic



Conversion to CNF

$$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$$

1. Eliminate \Leftrightarrow , replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$.

$$(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$$

2. Eliminate \Rightarrow , replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \vee \beta$.

$$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg(P_{1,2} \vee P_{2,1}) \vee B_{1,1})$$

3. Move \neg inwards using de Morgan's rules and double-negation:

$$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge ((\neg P_{1,2} \wedge \neg P_{2,1}) \vee B_{1,1})$$

4. Apply distributivity law (\vee over \wedge) and flatten:

$$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg P_{1,2} \vee B_{1,1}) \wedge (\neg P_{2,1} \vee B_{1,1})$$

Resolution algorithm

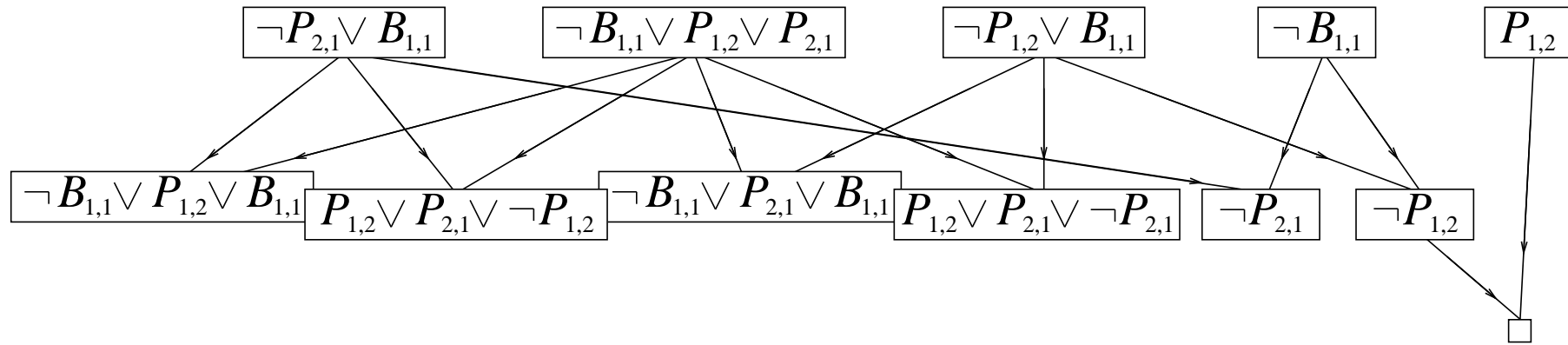
Proof by contradiction, i.e., show $KB \wedge \neg\alpha$ unsatisfiable

```
function PL-RESOLUTION( $KB, \alpha$ ) returns true or false
  inputs:  $KB$ , the knowledge base, a sentence in propositional logic
            $\alpha$ , the query, a sentence in propositional logic

   $clauses \leftarrow$  the set of clauses in the CNF representation of  $KB \wedge \neg\alpha$ 
   $new \leftarrow \{ \}$ 
  loop do
    for each  $C_i, C_j$  in  $clauses$  do
       $resolvents \leftarrow$  PL-RESOLVE( $C_i, C_j$ )
      if  $resolvents$  contains the empty clause then return true
       $new \leftarrow new \cup resolvents$ 
  if  $new \subseteq clauses$  then return false
   $clauses \leftarrow clauses \cup new$ 
```

Resolution example

$$KB = (B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})) \wedge \neg B_{1,1} \quad \alpha = \neg P_{1,2}$$



Summary

Logical agents apply **inference** to a **knowledge base** to derive new information and make decisions

Basic concepts of logic:

- **syntax**: formal structure of **sentences**
- **semantics**: **truth** of sentences wrt **models**
- **entailment**: necessary truth of one sentence given another
- **inference**: deriving sentences from other sentences
- **soundness**: derivations produce only entailed sentences
- **completeness**: derivations can produce all entailed sentences

Forward, backward chaining are linear-time, complete for Horn clauses

Resolution is complete for propositional logic