THE UNIVERSITY OF
WAIKATO
Te Whare Wānanga o Waikato

2007 B SEMESTER EXAMINATIONS

DEPARTMENT	Department of Computer Science
PAPER TITLE	Reasoning about Programs
TIME ALLOWED	Three Hours
NUMBER OF QUESTIONS IN PAPER	Eleven
NUMBER OF QUESTIONS TO BE ANSWERED	Eleven
VALUE OF EACH QUESTION	The value of each question is indicated. The total number of marks achievable is 100.
GENERAL INSTRUCTIONS	Answer ALL ELEVEN questions.
SPECIAL INSTRUCTIONS	None
CALCULATORS PERMITTED	No

Section A. Propositional and Predicate Logic

1.

a) Write down the →-elimination rule (→-elim) for natural deduction as given in the lectures.

b) What other name is frequently given to this rule?

c) Write down the →-introduction rule (→-intro) for natural deduction as given in the lectures.

d) Explain how to use the →-introduction rule.

e) Can there be a sound deductive system that does not use at least one of these two proof rules?

(2+2+2+2+2 marks)

2. Consider a language of predicate logic with the constant symbol leo, unary predicate symbols lion, zebra, and a binary predicate symbol eats. The intended meaning of the predicate symbols is as follows.

- $\mathsf{lion}(x)$ means that x is a lion.
- $\mathsf{zebra}(x)$ means that x is a zebra.
- $\mathsf{eats}(x, y)$ means that x likes to eat y.

Translate the following sentences into predicate logic, using only the above predicate symbols and the constant symbol leo.

a) Leo is a lion.

b) There exists a lion.

c) Nothing is both a lion and a zebra.

d) Some lions like to eat some zebras.

e) Some lions only like to eat zebras.

(2+2+2+2+2 marks)

3. Show that the following arguments are correct in propositional logic, using the system of natural deduction. List each proof step in a numbered line of its own, and indicate for each line how it is obtained. Use boxes to indicate the scope of assumptions.

a) **Premise 1:** $A \to C$
Premise 2: $B \to D$
Conclusion: $A \land B \to C \land D$

b) **Premise:** $A \land B \to C$
Conclusion: $B \land \neg C \to \neg A$

(5+5 marks)

4. Show that the following arguments are correct in predicate logic, using the system of natural deduction. List each proof step in a numbered line of its own, and indicate for each line how it is obtained. Use boxes to indicate the scope of assumptions and introduced constants.

a) **Premise 1:** $\forall x\ \mathsf{white}(x)$
Premise 2: $\forall x\ \mathsf{black}(x)$
Conclusion: $\forall x\ (\mathsf{white}(x) \wedge \mathsf{black}(x))$

b) **Premise 1:** $\mathsf{wabe}(\mathsf{b})$
Premise 2: $\forall x\ \mathsf{mimsy}(x)$
Premise 3: $\forall x\ ((\mathsf{brillig}(x) \vee \mathsf{tove}(x)) \rightarrow \exists y\ (\mathsf{gyre}(x,y) \wedge \mathsf{slithy}(y)))$
Premise 4: $\forall x\ (\mathsf{mimsy}(x) \rightarrow \mathsf{brillig}(x))$
Conclusion: $\exists x\, \exists y\ (\mathsf{gyre}(x,y) \wedge \mathsf{brillig}(y))$

(5+5 marks)

5. a) Translate the following predicate logic formula into Skolem Normal Form and extract a set of clauses from the result. Write each step in a line of its own and indicate clearly how it is obtained.

$$
\begin{array}{ll}
 & \forall x\, \forall y\, \forall z\ (\mathsf{p}(x,y) \wedge \mathsf{p}(y,z) \rightarrow \mathsf{p}(x,z)) \\
\wedge & \forall x\, \forall y\ (\mathsf{p}(x,y) \leftrightarrow \mathsf{p}(y,x)) \\
\wedge & \forall x\, \exists y\ \ \mathsf{p}(x,y)
\end{array}
$$

b) Use the method of resolution to prove that the following clauses are unsatisfiable together. Write each step in a numbered line of its own, showing clearly how it has been obtained and which substitution has been used.

$$
\begin{array}{l}
\{\neg\, \mathsf{p}(x,y), \neg\, \mathsf{p}(y,z), \mathsf{p}(x,z)\} \\
\{\neg\, \mathsf{p}(x,y), \mathsf{p}(y,x)\} \\
\{\mathsf{p}(x, \mathsf{f}(x))\} \\
\{\neg\, \mathsf{p}(\mathsf{f}(\mathsf{f}(x)), x)\}
\end{array}
$$

c) It has been elaborated in the lectures that the resolution theorem proving method is *refutation-complete* but not *complete*. Please explain briefly what this means.

(4+4+2 marks)

6. Given the definitions of the "divides" relation and the modulo operator for non-negative integers,

$$(\mathbf{DIV}) \quad \forall x{:}\,\mathbb{N}, y{:}\,\mathbb{N}\ (x \mid y \leftrightarrow \exists n{:}\,\mathbb{N}\ y = n \cdot x)$$

$$(\mathbf{MOD}) \quad \forall x{:}\,\mathbb{N}, y{:}\,\mathbb{N}\ (y \neq 0 \rightarrow x \bmod y < y \wedge \exists q{:}\,\mathbb{N}\ x = q \cdot y + x \bmod y)$$

and given the following properties of the greatest common divisor (gcd) of two non-negative integers,

$$(\mathbf{GCD}_{\text{divides}}) \quad \forall x{:}\,\mathbb{N}, y{:}\,\mathbb{N}\ (\gcd(x,y) \mid x \wedge \gcd(x,y) \mid y)$$

$$(\mathbf{GCD}_{\max}) \quad \forall d{:}\,\mathbb{N}, x{:}\,\mathbb{N}, y{:}\,\mathbb{N}\ (d \mid x \wedge d \mid y \rightarrow \gcd(x,y) \geq d)$$

$$(\mathbf{GCD}_{\text{eq}}) \quad \forall v{:}\,\mathbb{N}, w{:}\,\mathbb{N}, x{:}\,\mathbb{N}, y{:}\,\mathbb{N}\ (\forall d{:}\,\mathbb{N}\ (d \mid v \wedge d \mid w \leftrightarrow d \mid x \wedge d \mid y) \rightarrow \gcd(v,w) = \gcd(x,y))$$

write a proof in English, explaining why the following property of greatest common divisors is true.

$$(\mathbf{GCD}_{\text{mod}}) \quad \forall x{:}\,\mathbb{N}, y{:}\,\mathbb{N}\ (y \neq 0 \rightarrow \gcd(x,y) = \gcd(x \bmod y, y))$$

(10 marks)

Section B. Hoare Logic

7. Let C be a piece of code with P, D conditions. Suppose that the following Hoare triples and implications are correct:

$$(\mathbf{1}) \quad \{D \wedge P\}\ C\ \{P\}$$
$$(\mathbf{2}) \quad \{\neg D \wedge P\}\ C\ \{Q\}$$
$$(\mathbf{3}) \quad Q \ \rightarrow\ P \wedge D$$

Give proofs of the following triples, stating the rule(s) used.

a) $\{\neg D \wedge P\}\ C\ \{P\}$

b) $\{P\}\ C\ \{P\}$

c) $\{\neg D \wedge P\}\ C; C\ \{P\}$

d) $\{P\}$ `while` $\neg D$ `do` C $\{D \wedge P\}$

(1+2+1+2 marks)

8. Prove correctness for the following code, stating all rules used.

$\{\}$ `if` $a > b$ `then` $c := a - b$ `else` $c := 1$ $\{c > 0\}$

(4 marks)

9. a) Given that `n` is a positive integer, prove that the following code has empty precondition and postcondition $\{\, \texttt{sum} = \texttt{n}^2 \,\}$.

```
sum := 0;
i := 1;
while (i <= n) do
    begin
        sum := sum + 2*i - 1;
        i := i + 1
    end
```

b) Prove termination of the code in a).

(10+3 marks)

10. Prove the correctness of the following Hoare triple.

$$\{a[i] \geq 1,\ a[j] > 0\}\ a[i] := a[i] * a[j]\ \{a[i] \geq a[j],\ a[j] > 0\}.$$

(10 marks)

11. a) In Propositional Hoare Logic (PHL), what meanings do we give to a condition P and a piece of code C?

b) Prove that the precondition strengthening rule is valid under the standard relational interpretation of PHL.

c) Which of the usual Hoare logic rules does not carry over to PHL?

d) Give an example of a Hoare triple of PHL which is valid under the relational interpretation but which cannot be deduced from the rules of the logic.

(2+3+1+1 marks)