Symbols of Propositional Logic

Definition:

We denote by Σ the set of all propositional variables.

$$\Sigma = \{ p, p_1, p_2, \ldots, q, r, \ldots \}$$

Normally we assume:

$$true, false, \neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow, (,) \notin \Sigma$$

Definition of Well-Formed Formulas

Definition:

Let Σ be a set of propositional variables.
The set $\mathbf{WFF}_\Sigma$ of well-formed formulas of propositional logic for Σ is defined recursively:

- For each $p \in \Sigma$, we have $p \in \mathbf{WFF}_\Sigma$.
- If $A \in \mathbf{WFF}_\Sigma$ and $B \in \mathbf{WFF}_\Sigma$, then also
  $true, false \in \mathbf{WFF}_\Sigma$,
  $\neg A, (A \wedge B), (A \vee B), (A \oplus B), (A \rightarrow B), (A \leftrightarrow B) \in \mathbf{WFF}_\Sigma$.

Some Well-Formed Formulas

$$\neg\neg\neg p$$

$$(p \wedge q)$$

$$(((p \wedge q) \oplus r) \wedge s)$$

$$((p \wedge q) \rightarrow (q \wedge p))$$

$$(p \wedge ((q \leftrightarrow r) \wedge \neg p))$$

Omitting Brackets

- Binding priorities of operators in decreasing order:
  $\neg \quad \wedge \quad \vee \quad \oplus \quad \rightarrow \quad \leftrightarrow$
- The operators $\wedge$, $\vee$, $\oplus$, and $\leftrightarrow$ are understood to bind to the left, i.e.,
  $p \wedge q \wedge r$ is the same as $(p \wedge q) \wedge r$
- The outermost pair of brackets can always be deleted.

Syntax and Semantics

Syntax

The rules how formulas can be constructed.

- How well-formed formulas look like.

Semantics

The meaning of formulas.

- How well-formed formulas are interpreted.

Truth Tables for Formulas

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \to q$	$\neg(p\wedge q) \vee (p\to q)$
F	**F**	**F**	**T**	**T**	**T**
F	**T**	**F**	**T**	**T**	**T**
T	**F**	**F**	**T**	**F**	**T**
T	**T**	**T**	**F**	**T**	**T**

A particular assignment of truth values to propositional variables is an **interpretation**.

Interpretations

Definition:

Let Σ be a set of propositional variables.
An **interpretation** for Σ is a function

$$I : \Sigma \to \{\mathbf{T}, \mathbf{F}\}$$

Note

I can be extended to a map

$$I : \mathbf{WFF}_{\Sigma} \to \{\mathbf{T}, \mathbf{F}\}$$

Recursive Definition of I

$I(p) = I(p)$ if $p \in \Sigma$

$I(true) = \mathbf{T}$
$I(false) = \mathbf{F}$

$I(\neg A) = \mathbf{T}$ if $I(A) = \mathbf{F}$
$I(\neg A) = \mathbf{F}$ if $I(A) = \mathbf{T}$

$I(A\wedge B) = \mathbf{T}$ if $I(A) = \mathbf{T}$ and $I(B) = \mathbf{T}$
$I(A\wedge B) = \mathbf{F}$ if $I(A) = \mathbf{F}$ or $I(B) = \mathbf{F}$

Recursive Definition of I

$I(A\vee B) = \mathbf{T}$ if $I(A) = \mathbf{T}$ or $I(B) = \mathbf{T}$
$I(A\vee B) = \mathbf{F}$ if $I(A) = \mathbf{F}$ and $I(B) = \mathbf{F}$

$I(A\oplus B) = \mathbf{T}$ if $I(A) \neq I(B)$
$I(A\oplus B) = \mathbf{F}$ if $I(A) = I(B)$

$I(A\to B) = \mathbf{T}$ if $I(A) = \mathbf{F}$ or $I(B) = \mathbf{T}$
$I(A\to B) = \mathbf{F}$ if $I(A) = \mathbf{T}$ and $I(B) = \mathbf{F}$

$I(A\leftrightarrow B) = \mathbf{T}$ if $I(A) = I(B)$
$I(A\leftrightarrow B) = \mathbf{F}$ if $I(A) \neq I(B)$

Finding the Logical Status of a Formula

1. Construct truth table
2. Check each line …

All lines **T** ⇝ valid

All lines **F** ⇝ unsatisfiable

Otherwise ⇝ satisfiable but not valid

Example

$$\neg p \to q \wedge \neg r$$

p	q	r	$\neg p$	$\neg r$	$q \wedge \neg r$	$\neg p \to q \wedge \neg r$
F	**F**	**F**	**T**	**T**	**F**	**F**
F	**F**	**T**	**T**	**F**	**F**	**F**
F	**T**	**F**	**T**	**T**	**T**	**T**
F	**T**	**T**	**T**	**F**	**F**	**F**
T	**F**	**F**	**F**	**T**	**F**	**T**
T	**F**	**T**	**F**	**F**	**F**	**T**
T	**T**	**F**	**F**	**T**	**T**	**T**
T	**T**	**T**	**F**	**F**	**F**	**T**

Models

$$\neg p \to q \wedge \neg r$$

p	q	r	$\neg p$	$\neg r$	$q \wedge \neg r$	$\neg p \to q \wedge \neg r$
F	F	F	T	T	F	F
F	F	T	T	F	F	F
F	**T**	**F**	**T**	**T**	**T**	**T**
					F	F

I such that

$I(p) = $ **F**, $I(q) = $ **T**, $I(r) = $ **F**

is a **model** of the formula $\neg p \to q \wedge \neg r$.

What is a Model?

Definition:

Let $A \in$ **WFF**$_\Sigma$ be a formula.

An interpretation $I: \Sigma \to \{$**T**, **F**$\}$ is called a **model** for A if

$$I(A) = \text{T}.$$

Models and Logical Status

A formula A is …

- **Valid**
  if every interpretation is a model for A.
- **Satisfiable**
  if A has a model.
- **Unsatisfiable**
  if A has no model.

Logical Equivalence

Definition:

Let $A, B \in$ **WFF**$_\Sigma$ be two formulas.

A and B are said to be **logically equivalent**, if A and B have the same truth values for all possible interpretations,

i.e., if $I(A) = I(B)$

for every interpretation $I: \Sigma \to \{$**T**,**F**$\}$.

Checking Logical Equivalence

To check whether two formulas A and B are logically equivalent:

1. Construct truth tables for A and B.
2. If the truth values of A and B are the same in all rows then A and B are logically equivalent.

Example

Checking whether:

$p \leftrightarrow q$ is logically equivalent to $(p \to q) \wedge (q \to p)$.

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
F	**F**				
F	**T**				
T	**F**				
T	**T**				

Logical Consequence

Definition:

Let A, $B \in$ **WFF**$_\Sigma$ be two formulas.

B is said to be a **logical consequence** of A, if every model for A also is a model for B.

Checking Logical Consequence

To check whether B is a logical consequence of A:

1. Construct truth tables for A and B.
2. If every line showing the truth value **T** for A also shows the truth value **T** for B, then B is a logical consequence of A.
3. Lines showing the truth value **F** for A do not matter.

Example

Is $p \vee q$ is a logical consequence $p \oplus q$?

p	q	$p \oplus q$	$p \vee q$
F	F	F	F
F	**T**	**T**	**T**
T	**F**	**T**	**T**
T	T	F	T

$p \oplus q$ is true, so $p \vee q$ must also be true.

$p \oplus q$ is false, so $p \vee q$ does not matter.

Thus, $p \vee q$ is a logical consequence $p \oplus q$.

Example

Is $p \oplus q$ is a logical consequence $p \vee q$?

p	q	$p \vee q$	$p \oplus q$
F	F	F	F
F	T	T	T
T	F	T	T
T	**T**	**T**	**F**

✗ $p \vee q$ is true, but $p \oplus q$ is false.

$p \oplus q$ is **not** a logical consequence $p \vee q$.

Relationship to Validity

Let A, $B \in$ **WFF**$_\Sigma$ be two formulas.

- A and B are logically equivalent if and only of $A \leftrightarrow B$ is a *valid* formula.
- B is a logical consequence of A if and only if $A \rightarrow B$ is a *valid* formula.

Reading

Huth & Ryan:
Section 1.3–1.4.1
pp. 31–40