COMP340-08B Reasoning About Programs

4. Normal Forms

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Disjunctive Normal Form (DNF)

Definition:

A formula is in **disjunctive normal form** if it is a disjunction of conjunctions of **literals**.

p or ¬*p*

conjunction *conjunction* *conjunction*

$$\underbrace{(p \wedge q)}_{} \vee \underbrace{(\neg p \wedge r)}_{} \vee \underbrace{(q \wedge r \wedge \neg s)}_{}$$

disjunction

Lots of Connectives

$\leftrightarrow$ $\oplus$ $\rightarrow$

$\vee$ $\neg$ $\wedge$

Do we really need all these connectives?

Conjunctive Normal Form (CNF)

Definition:

A formula is in **conjunctive normal form** if it is a conjunction of disjunctions of literals.

Clause

disjunction *disjunction* *disjunction*

$$\underbrace{(p \vee q)}_{} \wedge \underbrace{(\neg p \vee r)}_{} \wedge \underbrace{(q \vee r \vee \neg s)}_{}$$

conjunction

Laws of Equivalence

Definition Laws:

- $A \rightarrow B$ is equivalent to $\neg A \vee B$.
- $A \leftrightarrow B$ is equivalent to $(A \rightarrow B) \wedge (B \rightarrow A)$.
- $A \oplus B$ is equivalent to $(A \wedge \neg B) \vee (B \wedge \neg A)$.

DeMorgan's Laws:

- $\neg(A \wedge B)$ is equivalent to $\neg A \vee \neg B$.
- $\neg(A \vee B)$ is equivalent to $\neg A \wedge \neg B$.

Properties of CNF and DNF

- Any propositional formula can be converted into CNF or DNF using laws of equivalence.
- Why are CNF and DNF important?
  - Easy to understand and analyze.
  - Can be easily converted into a circuit.

Algorithm to Construct CNF (or DNF)

1. Apply the **Definitions of →, ↔, and ⊕** to remove all occurrences of these connectives.
2. Move negation inwards using
   - **Double Negation** or
   - **De Morgan's Laws**
3. Use the **Distributivity Law** to reduce the scope of $\wedge$ (or $\vee$).

Finding a CNF: an Example

Find CNF equivalent to $\neg((p \rightarrow q) \wedge r)$:

$\neg((p \rightarrow q) \wedge r)$	
$\neg((\neg p \vee q) \wedge r)$	(*Definition of* $\rightarrow$)
$\neg(\neg p \vee q) \vee \neg r$	(*De Morgan*)
$(\neg\neg p \wedge \neg q) \vee \neg r$	(*De Morgan*)
$(p \wedge \neg q) \vee \neg r$	(*Double Negation*)
$(p \vee \neg r) \wedge (\neg q \vee \neg r)$	(*Distributivity*)

Minterms

Definition:

A **minterm** is a conjunction of literals in which each propositional variable occurs exactly once.

Example:

$(\neg p \wedge q \wedge \neg r)$ is a minterm.
(if we have only the variables p, q, and r.)

Full Disjunctive Normal Form

Definition:

A formula is in **full disjunctive normal form** if it is a disjunction minterms.

$$(\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge q \wedge r)$$

Properties of Full DNF

- Every formula can be converted into full DNF.
- Full DNF can be constructed from truth tables.
- Two formulas are logically equivalent if and only if their full DNFs are equal (up to reordering of subterms).

Reading

Huth & Ryan:
Section 1.5.1–1.5.2
pp. 54–65