COMP340-08B Reasoning About Programs

6. Semantics of Predicate Logic

Robi Malik

THE UNIVERSITY OF WAIKATO
DEPARTMENT OF COMPUTER SCIENCE
TARI ROROHIKO

What is Semantics?

$\forall x\, (\mathrm{nurk}(x) \wedge \exists y\, \mathrm{lop}(x, y) \rightarrow \mathrm{meve}(x))$

Semantics is about:

- Giving *meaning* to a formula.
- Given a world, determine whether a formula is *true* in that world.

From Propositional to Predicate Logic

- In propositional logic, possible worlds are characterised the interpretation of **propositional variables**.

"Socrates is a man" → "Socrates is mortal"

- In predicate logic, we have to consider **objects** in addition.

$\forall x\, (\mathrm{man}(x) \rightarrow \mathrm{mortal}(x))$

$\mathrm{man}(\mathrm{socrates}) \rightarrow \mathrm{mortal}(\mathrm{socrates})$

What is an Interpretation?

Characterising a possible world …

$\mathrm{nurk}(\mathrm{dork})$

$\forall x\, (\mathrm{nurk}(x) \wedge \exists y\, \mathrm{lop}(x, y) \rightarrow \mathrm{meve}(x))$

An interpretation must include:

- The set of objects that exist in the world.
- The meaning of the predicate symbols nurk, lop, and meve.
- The meaning of the constant symbol dork.

Definition of an Interpretation

Definition:
Let $\Sigma = (\mathcal{P}, \mathcal{F}, \mathcal{V})$ be a signature.
An **interpretation** for Σ is a pair $I = (\mathbf{D}, \mathbf{F})$, where

- $\mathbf{D}$ is a set, called the **domain of discourse** of I.
- $\mathbf{F}$ is a function, assigning to each …
  - n-ary predicate symbol $\mathrm{p} \in \mathcal{P}$ — an n-ary relation $\mathrm{p}^I \subseteq \mathbf{D}^n$.
  - constant symbol $\mathrm{a} \in \mathcal{F}$ — an element $\mathrm{a}^I \in \mathbf{D}$.
  - n-ary function symbol $\mathrm{f} \in \mathcal{F}$, $n \geq 1$ — an n-ary function $\mathrm{f}^I\colon \mathbf{D}^n \rightarrow \mathbf{D}$.
  - variable symbol $x \in \mathcal{V}$ — an element $x^I \in \mathbf{D}$.

An Example

$\forall x\, (\mathrm{nurk}(x) \wedge \exists y\, \mathrm{lop}(x, y) \rightarrow \mathrm{meve}(x))$

Domain of discourse: $\mathbf{D} = \{a, b, c, d\}$

Interpretation of predicate symbols:

	nurk^I	meve^I
a	T	F
b	T	T
c	T	F
d	T	T

lop^I	a	b	c	d
a	F	F	F	F
b	T	T	F	T
c	F	F	F	F
d	T	F	F	F

Completing the Example

$$\text{nurk}(\text{dork})$$
$$\forall x\ (\text{nurk}(x) \wedge \exists y\ \text{lop}(x, y) \rightarrow \text{meve}(x))$$

Domain of discourse:

$\mathbf{D} = \{$a, b, c, d$\}$

Interpretation of Predicate symbols:

nurk^I, lop^I, meve^I (as given by the table)

Interpretation of Function and Constant symbols:

$\text{dork}^I = $ d

Assignment of Variable symbols:

$x^I = $ a, $y^I = $ b

Extending an Interpretation

An interpretation consists of

- Domain of discourse
- Interpretation of predicate symbols
- Interpretation of function symbols
- Variable assignment

Given these components, we can determine

- the interpretation of terms in $\mathbf{TERM}_\Sigma$;
- the truth value of formulas in $\mathbf{WFF}_\Sigma$.

Interpreting Terms

Definition:

Let $\Sigma = (\mathcal{P}, \mathcal{F}, \mathcal{V})$ be a signature, and let $I = (\mathbf{D}, \mathbf{F})$ be an interpretation for Σ.

Then we consider I as a map $I : \mathbf{TERM}_\Sigma \rightarrow \mathbf{D}$ as follows:

- $I\,(x) = x^I$ for each variable symbol $x \in \mathcal{V}$.
- $I\,(\text{a}) = \text{a}^I$ for each constant symbol $\text{a} \in \mathcal{F}$.
- $I\,(\text{f}(T_1,\ldots,T_n)) = \text{f}^I(I(T_1),\ldots,I(T_n))$
  for each n-ary function symbol $\text{f} \in \mathcal{F}$.

I assigns to each term in $\mathbf{TERM}_\Sigma$ an object in $\mathbf{D}$.

Interpreting Formulas

Definition:

Let $\Sigma = (\mathcal{P}, \mathcal{F}, \mathcal{V})$ be a signature, and let $I = (\mathbf{D}, \mathbf{F})$ be an interpretation for Σ.

Then we consider I as a map $I : \mathbf{WFF}_\Sigma \rightarrow \{\mathbf{T}, \mathbf{F}\}$ as follows:

- $I\,(\text{p}(T_1,\ldots,T_n)) = \text{p}^I(I(T_1),\ldots,I(T_n))$
  for each n-ary predicate symbol $\text{p} \in \mathcal{P}$.
- $I\,(true) = \mathbf{T}$
- $I\,(\,false) = \mathbf{F}$

…

Interpreting Formulas continued

Definition (continued):

…

- $I\,(\neg A) = \mathbf{T}$ if and only if $I(A) = \mathbf{F}$
- $I\,(A \wedge B) = \mathbf{T}$ if and only if $I(A) = I(B) = \mathbf{T}$
- $I\,(A \vee B) = \mathbf{T}$ if $I(A) = \mathbf{T}$ or $I(B) = \mathbf{T}$
- $I\,(A \rightarrow B) = \mathbf{T}$ if $I(A) = \mathbf{F}$ or $I(B) = \mathbf{T}$
- $I\,(A \leftrightarrow B) = \mathbf{T}$ if $I(A) = I(B)$
- $I\,(A \oplus B) = \mathbf{T}$ if $I(A) \neq I(B)$
- $I\,(\forall x\ A) = \mathbf{T}$ if $I[c/x](A) = \mathbf{T}$ for every $c \in \mathbf{D}$.
- $I\,(\exists x\ A) = \mathbf{T}$ if $I[c/x](A) = \mathbf{T}$ for some $c \in \mathbf{D}$.

I assigns a truth value to each formula in $\mathbf{WFF}_\Sigma$.

Changing the Variable Assignment

Definition:

Let $\Sigma = (\mathcal{P}, \mathcal{F}, \mathcal{V})$ be a signature, and let $I = (\mathbf{D}, \mathbf{F})$ be an interpretation for Σ. Furthermore, let $x \in \mathcal{V}$ be a variable symbol, and let $c \in \mathbf{D}$. Then we define the interpretation

$$I\,[c/x] = (\mathbf{D}, \mathbf{F}')$$

such that

$$\mathbf{F}'(A) = \begin{cases} c, & \text{if } A = x; \\ \mathbf{F}(A), & \text{otherwise.} \end{cases}$$

Another Example

$$\forall x\ (\text{nurk}(x) \wedge \exists y\ \text{lop}(x, y) \rightarrow \text{meve}(x))$$

Formula is **true** for this interpretation.

Domain of discourse:

$\mathbf{D}$ = {Socrates}

Interpretation of predicates:

nurk^{I}(Socrates) = **T**

lop^{I}(Socrates, Socrates) = **T**

meve^{I}(Socrates) = **T**

Interpretation of variables:

x^{I} = Socrates, y^{I} = Socrates

Standard Concepts of Logic

Given the notion of interpretations, we can now introduce the following concepts known from propositional logic:

- Model
- Satisfiability
- Validity
- Logical consequence
- Logical equivalence

Models, Satisfiability and Validity

Definition:

Let A be a predicate logic formula.

- An interpretation $I = (\mathbf{D}, \mathbf{F})$ is called a **model** for A, if $I(A) = \mathbf{T}$.
- The formula A is called **valid**, if every interpretation is a model for A.
- The formula A is called **satisfiable**, if there exists a model for A.
  Otherwise A is called **unsatisfiable**.

Logical Consequence and Equivalence

Let A and B be two predicate logic formulas.

- A is a **logical consequence** of B, if every model for B also is a model for A.
- A and B are **logically equivalent**, if A is a logical consequence of B and vice versa.

Let Φ be a set of predicate logic formulas.

- $I = (\mathbf{D}, \mathbf{F})$ is a **model** for Φ, if I is a model for every formula $A \in \Phi$.
- A is a **logical consequence** of Φ, if every model for Φ also is a model for A.

Valid Formulas

Some formulas are true in all possible worlds!

$$\forall x\ (\text{nurk}(x) \vee \neg\text{nurk}(x))$$

$$\forall x\ \text{happy}(x) \leftrightarrow \forall y\ \text{happy}(y)$$

$$\forall x\ (\text{p}(x) \wedge \text{q}(x)) \leftrightarrow (\forall x\ \text{p}(x) \wedge \forall x\ \text{q}(x))$$

$$\forall x\ \text{p}(x) \leftrightarrow \neg\exists x\ \neg\text{p}(x)$$

$$\exists x\ \text{p}(x) \leftrightarrow \neg\forall x\ \neg\text{p}(x)$$

Reading

Huth & Ryan:
Section 2.4
pp. 122–131.