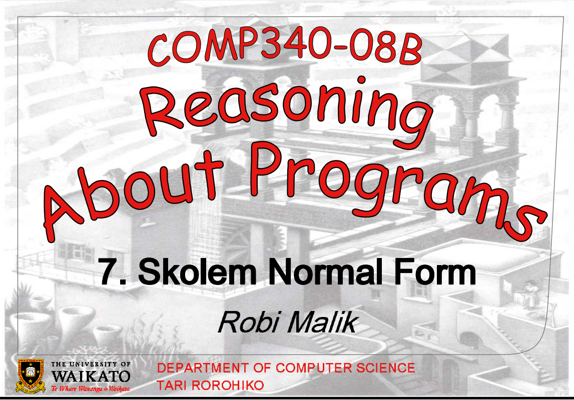

COMP340-08B Reasoning About Programs

7. Skolem Normal Form

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DeMorgan's Laws for Quantifiers

$\exists x\ A(x)$
is logically equivalent to
$\neg\forall x\ \neg A(x)$.

$\forall x\ A(x)$
is logically equivalent to
$\neg\exists x\ \neg A(x)$.

Laws of Equivalence for Quantifiers

Renaming Laws:

- $\forall x\ A(x)$ is equivalent to $\forall y\ A(y)$
- $\exists x\ A(x)$ is equivalent to $\exists y\ A(y)$

… provided $A(x)$ does not contain y
… and $A(y)$ does not contain x.

Scoping Laws:

- $\forall x\ (A * B(x))$ is equivalent to $A * \forall x\ B(x)$
- $\forall x\ (A * B(x))$ is equivalent to $A * \forall x\ B(x)$

… provided A does not contain free occurrences of x.
… where * can be either $\vee$ or $\wedge$.

Prenex Conjunctive Normal Form

Definition:

A predicate logic formula is in **prenex conjunctive normal form** if it consists of a **prefix** of universal and existential quantifiers followed by a quantifier-free **matrix** in conjunctive normal form.

$$\underbrace{\forall x_1 \exists x_2 \ldots \forall x_n}_{\text{Prefix of quantifiers}} \underbrace{((\mathrm{p}(x_1) \vee \mathrm{q}(x_2,x_3) \vee \ldots) \wedge \ldots}_{\text{Matrix in CNF}}$$

Algorithm to Construct Prenex Form

1. Apply the **Definitions of $\rightarrow$, $\leftrightarrow$, and $\oplus$** to remove all occurrences of these connectives.
2. Move negation inwards using **Double Negation** or **De Morgan's Laws**.
3. Rename bound variables
   $\rightarrow$ Each name may occur only one.
4. Move quantifiers to front.
5. Use the **Distributivity Law** to increase the scope of $\wedge$.

Skolem Normal Form (SkNF)

Definition:

A predicate logic formula is in **Skolem Normal Form** if it consists of a **prefix** of universal quantifiers followed by a quantifier-free **matrix** in conjunctive normal form.

$$\underbrace{\forall x_1 \forall x_2 \ldots \forall x_n}_{\text{Prefix universal quantifiers}} \underbrace{(\mathrm{p}(x_1) \vee \mathrm{q}(x_2,x_3) \vee \ldots) \wedge \ldots}_{\text{Matrix in CNF}}$$

Eliminating Existential Quantifiers

DeMorgan's Law:

$\exists x\, A$ is logically equivalent to $\neg\forall x\, \neg A$

But …

This does not allow us to rewrite

$$\exists x_1\, \exists x_2 \ldots A$$

into

$$\forall x_1\, \forall x_2 \ldots A'$$

Examples of Skolem Functions

$\exists x$ killer(x)

→ killer(**mr_x**) (Constant symbol)

$\forall x\, \exists y$ son_of(x, y)

→ $\forall x$ son_of(x, **father**(x)) (Unary function symbol)

$\forall x\, \exists y$ married(x, y)

→ $\forall x$ married(x, **spouse**(x))

$\forall x$ (nat(x) $\rightarrow \exists y$ greater(y, x))

→ $\forall x$ (nat(x) $\rightarrow$ greater(**next**(x), x))

Skolemization Algorithm

1. Replace all occurrences of $\rightarrow$, $\leftrightarrow$, and $\oplus$.
2. Move negation inwards.
3. Standardize variables apart.
4. Replace existential quantifiers by **Skolem Functions**.
5. Move universal quantifiers to front.
6. Use the Distributivity Law.

Introducing Skolem Functions

4. Replace the existential quantifier in subformula $\exists x\, A$ by a Skolem Function:
   a) Find all universal quantifiers, in whose scope the subformula $\exists x\, A$ occurs:
      Let $y_1,\ldots,y_n$ be the variables quantified by those quantifiers.
   b) Create a new n-ary function symbol f.
   c) Replace all occurrences of x by $\mathrm{f}(y_1,\ldots,y_n)$.
   d) Remove the existential quantifier for x.

Example

$$\forall x\, (\forall y\, \mathrm{p}(x,y) \rightarrow \neg\forall y\, (\mathrm{q}(x,y) \rightarrow \mathrm{p}(x,y)))$$
$$\forall x\, (\neg\forall y\, \mathrm{p}(x,y) \vee \neg\forall y\, (\neg\mathrm{q}(x,y) \vee \mathrm{p}(x,y)))$$
$$\forall x\, (\exists y\, \neg\mathrm{p}(x,y) \vee \exists y\, \neg(\neg\mathrm{q}(x,y) \vee \mathrm{p}(x,y)))$$
$$\forall x\, (\exists y\, \neg\mathrm{p}(x,y) \vee \exists y\, (\neg\neg\mathrm{q}(x,y) \wedge \neg\mathrm{p}(x,y)))$$
$$\forall x\, (\exists y\, \neg\mathrm{p}(x,y) \vee \exists y\, (\mathrm{q}(x,y) \wedge \neg\mathrm{p}(x,y)))$$
$$\forall x\, (\exists y_1\, \neg\mathrm{p}(x,y_1) \vee \exists y_2\, (\mathrm{q}(x,y_2) \wedge \neg\mathrm{p}(x,y_2)))$$
$$\forall x\, (\neg\mathrm{p}(x,\mathrm{f}(x)) \vee (\mathrm{q}(x,\mathrm{g}(x)) \wedge \neg\mathrm{p}(x,\mathrm{g}(x)))$$
$$\forall x\, ((\neg\mathrm{p}(x,\mathrm{f}(x)) \vee \mathrm{q}(x,\mathrm{g}(x)) \wedge (\neg\mathrm{p}(x,\mathrm{f}(x)) \vee \neg\mathrm{p}(x,\mathrm{g}(x))))$$

Property of Skolem Normal Form

Every predicate logic formula A can be rewritten to a formula A' in Skolem Normal Form such that

A is **satisfiable**
if and only if
A' is **satisfiable**.

Note:

A and A' are **not necessarily equivalent**.