# COMP340-08B Reasoning About Programs

## 8. Natural Deduction

*Robi Malik*

THE UNIVERSITY OF WAIKATO Te Whare Wānanga o Waikato

DEPARTMENT OF COMPUTER SCIENCE
TARI ROROHIKO

## Natural Deduction

- A formal framework to derive proofs in propositional and predicate logic.
- Produces proofs that are similar in style to common (mathematical) reasoning.
- Sound and complete.
- Today: Only propositional logic.



## Elimination and Introduction

Natural deduction uses two types of inference rules for each connective:

- **Elimination rules:**
  Remove a connective from a formula.
- **Introduction rules:**
  Add a connective to a formula.



## Rules for $\wedge$

**$\wedge$-Elimination**

($\wedge$-elim-1) $\dfrac{A \wedge B}{A}$ ($\wedge$-elim-2) $\dfrac{A \wedge B}{B}$

**$\wedge$-Introduction**

($\wedge$-intro) $\dfrac{A \quad B}{A \wedge B}$



## Rules for $\rightarrow$

**$\rightarrow$-Elimination (Modus Ponens)**

($\rightarrow$-elim) $\dfrac{A \rightarrow B \quad A}{B}$

**$\rightarrow$-Introduction (Conditional Proof)**

($\rightarrow$-intro) $\dfrac{\dfrac{A}{B}}{A \rightarrow B}$



## Example Proof

**Premises:** $p,\ q,\ p \wedge q \rightarrow r \wedge s$

**Conclusion:** $s$

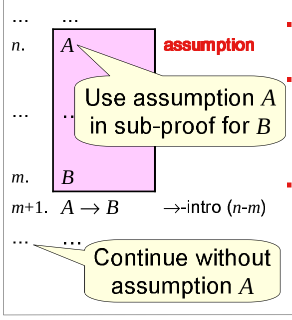
| | | |
|---|---|---|
| 1. | $p$ | premise |
| 2. | $q$ | premise |
| 3. | $p \wedge q \rightarrow r \wedge s$ | premise |
| 4. | $p \wedge q$ | $\wedge$-intro: 1,2 |
| 5. | $r \wedge s$ | $\rightarrow$-elim: 4,3 |
| 6. | $s$ | $\wedge$-elim-2: 5 |

## Remember

- A **proof** is a step-by-step demonstration that the conclusion follows from the premises.
- Each step is
  - numbered (for future references) and
  - justified by an appropriate rule.



## Assumed Premises




- Some rules introduce new premises.
- These temporary premises are called **assumed premises** or **assumptions**.
- They are only available in a sub-proof and must be **discharged**, i.e., forgotten when the sub-proof is completed.



## Conditional Proof

$$(\to\text{-intro})\ \frac{\frac{A}{B}}{A \to B}$$

In order to prove $A \to B$,
we assume that $A$ holds
and prove $B$ from this assumption.



## Rules for ∨

**∨-Introduction**

$$(\vee\text{-intro-1})\ \frac{A}{A \vee B} \qquad (\vee\text{-intro-2})\ \frac{B}{A \vee B}$$

**∨-Elimination (Proof by Cases)**

$$(\vee\text{-elim})\ \frac{A \vee B \quad \frac{A}{C} \quad \frac{B}{C}}{C}$$



## Example Using Conditional Proof

| | | |
|---|---|---|
| 1. | $p \to r$ | assumption |
| 2. | $p \wedge q$ | assumption |
| 3. | $p$ | ∧-elim-1: 2 |
| 4. | $r$ | →-elim: 1,3 |
| 5. | $p \wedge q \to r$ | →-intro: 2–4 |
| 6. | $(p \to r) \to (p \wedge q \to r)$ | →-intro: 1–5 |



## Proof by Cases: an Example

**Premise:** $(p \wedge s) \vee (q \wedge s)$
**Conclusion:** $s$

| | | |
|---|---|---|
| 1. | $(p \wedge s) \vee (q \wedge s)$ | premise |
| 2. | $p \wedge s$ | assumption |
| 3. | $s$ | ∧-elim-2: 2 |
| 4. | $q \wedge s$ | assumption |
| 5. | $s$ | ∧-elim-2: 4 |
| 6. | $s$ | ∨-elim: 1,2–3,4–5 |

## Rules for $\leftrightarrow$

**$\leftrightarrow$-Elimination**

(**$\leftrightarrow$-elim-1**) $\dfrac{A \leftrightarrow B}{A \rightarrow B}$ (**$\leftrightarrow$-elim-2**) $\dfrac{A \leftrightarrow B}{B \rightarrow A}$

**$\leftrightarrow$-Introduction**

(**$\leftrightarrow$-intro**) $\dfrac{A \rightarrow B \quad B \rightarrow A}{A \leftrightarrow B}$



## Example with Equivalence

| | | |
|---|---|---|
| 1. | $p \wedge q$ | assumption |
| 2. | $p$ | $\wedge$-elim-1: 1 |
| 3. | $q$ | $\wedge$-elim-2: 1 |
| 4. | $q \wedge p$ | $\wedge$-intro: 2,3 |
| 5. | $(p \wedge q) \rightarrow (q \wedge p)$ | $\rightarrow$-intro: 1–4 |
| 6. | $q \wedge p$ | assumption |
| 7. | $q$ | $\wedge$-elim-1: 6 |
| 8. | $p$ | $\wedge$-elim-2: 6 |
| 9. | $p \wedge q$ | $\wedge$-intro: 8,7 |
| 10. | $(q \wedge p) \rightarrow (p \wedge q)$ | $\rightarrow$-intro: 6–9 |
| 11. | $(p \wedge q) \leftrightarrow (q \wedge p)$ | $\leftrightarrow$-intro: 5,10 |



## Proving Equivalences

To establish an equivalence

$$A \leftrightarrow B$$

we have to prove two implications:

- $A \rightarrow B$ – Assume $A$ and prove $B$
- $B \rightarrow A$ – Assume $B$ and prove $A$



## Natural Deduction — Tips

1. Apply all possible elimination rules to all premises and assumptions

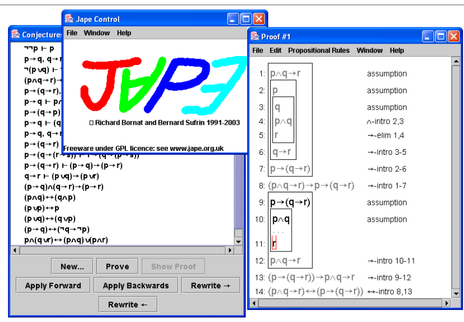
   *This gives more and more premises that will help you to complete the proof.*

2. Apply the appropriate introduction rule to the conclusion

   *This may produce new assumptions or conclusions – then repeat these two steps.*



## J∀P∃ – a Proof Assistant






## Reading

Huth & Ryan:
Section 1.2
pp. 5–31.