COMP340-08B Reasoning About Programs

9. Proof by Contradiction

Robi Malik

THE UNIVERSITY OF WAIKATO
DEPARTMENT OF COMPUTER SCIENCE
TARI ROROHIKO

Natural Deduction

A simple deductive system
for propositional and predicate logic.

- ***So far ...***

  10 rules for the connectives

  $\wedge \quad \vee \quad \rightarrow \quad \leftrightarrow$

- ***Today ...***

  3 rules for $\neg$

Rules for $\neg$

($false$-intro) $\dfrac{A \quad \neg A}{false}$

($false$-**elim**) $\dfrac{\neg A \quad false}{A}$

($\neg$-**intro**) $\dfrac{A \quad false}{\neg A}$

} *Proof by Contradiction*

Proof by Contradiction

If we want to prove $\neg A$:

1. Assume that A is true.
2. Using the assumption,
   prove a **contradiction**,
   i.e., prove that $false$ would follow.
3. Conclude that A cannot be true,
   i.e., A must be false.

Proof Rules Using Contradiction (1)

($\neg$-intro) $\dfrac{\begin{array}{c}A\\ \hline false\end{array}}{\neg A}$

In order to prove $\neg A$,
we assume that A holds and
show that this yields a contradiction.

Proof Rules Using Contradiction (2)

($false$-**elim**) $\dfrac{\begin{array}{c}\neg A\\ \hline false\end{array}}{A}$

In order to prove A,
we assume that A does not hold and
show that this yields a contradiction.

Proof by Contradiction: an Example

Premise: $p \to q$
Conclusion: $\neg q \to \neg p$

1.	$p \to q$	premise
2.	$\neg q$	assumption
3.	p	assumption
4.	q	$\to$-elim: 1,3
5.	*false*	*false*-intro: 4,2
6.	$\neg p$	$\neg$-intro: 3–5
7.	$\neg q \to \neg p$	$\to$-intro: 2–6

About Contradictions

Note:

- The formula *false* $\to$ A is valid.
- Any formula is a logical consequence of the formula *false*.
- If a set of formulas is unsatisfiable, any formula logically follows from it.
- If we can ever prove *false*, we can infer any formula from this.

Contraposition and Indirect Proof

- **Contraposition**

  The formulas $A \to B$ and $\neg B \to \neg A$ are logically equivalent.

  Contraposition of $A \to B$

- **Indirect Proof**

  Instead of proving $A \to B$
  we can also prove $\neg B \to \neg A$.
  This is sometimes easier.

Yet Another Example

1.	$\neg p \wedge \neg q$	premise
2.	$p \vee q$	assumption
3.	p	assumption
4.	$\neg p$	$\wedge$-elim-1: 1
5.	*false*	*false*-intro: 3,4
6.	q	assumption
7.	$\neg q$	$\wedge$-elim-2: 1
8.	*false*	*false*-intro: 6,8
9.	*false*	$\vee$-elim: 2,3-5,6-8
10.	$\neg(p \vee q)$	$\neg$-intro: 2–9

Another Example

1.	$p \wedge \neg p$	assumption
2.	$\neg q$	assumption
3.	p	$\wedge$-elim-1: 1
4.	$\neg p$	$\wedge$-elim-2: 1
5.	*false*	*false*-intro: 3,4
6.	q	*false*-elim: 2–5
7.	$(p \wedge \neg p) \to q$	$\to$-intro: 1–6

And One More ...

1.	$\neg(p \vee \neg p)$	assumption
2.	p	assumption
3.	$p \vee \neg p$	$\vee$-intro-1: 2
4.	*false*	*false*-intro: 3,1
5.	$\neg p$	$\neg$-intro: 2–4
6.	$\neg p$	assumption
7.	$p \vee \neg p$	$\vee$-intro-2: 2
8.	*false*	*false*-intro: 7,1
9.	p	*false*-elim: 6–8
10.	*false*	*false*-intro: 9,5
11.	$p \vee \neg p$	*false*-elim: 1–10