COMP340-08B Reasoning About Programs

10. Natural Deduction with Quantifiers

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Natural Deduction

- A sound and complete deductive system for propositional logic with
  - 13 inferences rules
- Extend it for predicate logic with
  - 4 additional inference rules

Natural Deduction Rules for ∀

∀-Elimination

(**∀-elim**) $\dfrac{\forall x\, A(x)}{A(t)}$ (Provided that $t \in$ **TERM**$_\Sigma$ does not contain any variables)

∀-Introduction

(**∀-intro**) $\dfrac{\dfrac{}{A(c)}}{\forall x\, A(x)}$ (Provided that c is a *new* constant symbol)

An Example Proof

Premises: $\forall x\,(\mathrm{old}(x) \rightarrow \mathrm{wise}(x))$, $\forall x\, \mathrm{old}(x)$

Conclusion: $\forall x\, \mathrm{wise}(x)$

1.	$\forall x\,(\mathrm{old}(x) \rightarrow \mathrm{wise}(x))$	premise
2.	$\forall x\, \mathrm{old}(x)$	premise
	(Let c be an arbitrary element.)	
3.	$\mathrm{old}(c)$	∀-elim: 2
4.	$\mathrm{old}(c) \rightarrow \mathrm{wise}(c)$	∀-elim: 1
5.	$\mathrm{wise}(c)$	→-elim: 4, 5
6.	$\forall x\, \mathrm{wise}(x)$	∀-intro: 3–5

New Constant Symbols

This symbol must not have been used before.

1.	…	
2.	…	
	(Let c be an arbitrary element.)	
3.	…	
4.	…	
5.	…	
6.	…	∀-intro: 3–5

Natural Deduction Rules for ∃

∃-Introduction

(**∃-intro**) $\dfrac{A(t)}{\exists x\, A(x)}$

∃-Elimination

(**∃-elim**) $\dfrac{\exists x\, A(x) \quad \dfrac{A(c)}{B}}{B}$ (Provided c is a *new* constant symbol)

An Example Proof

Premises: $\forall x\,(\text{old}(x) \rightarrow \text{wise}(x)),\ \exists x\,\text{old}(x)$
Conclusion: $\exists x\,\text{wise}(x)$

1.	$\forall x\,(\text{old}(x) \rightarrow \text{wise}(x))$	premise
2.	$\exists x\,\text{old}(x)$	premise
	(Let c be the object satisfying $\text{old}(x)$.)	
3.	$\text{old}(c)$	assumption
4.	$\text{old}(c) \rightarrow \text{wise}(c)$	$\forall$-elim: 1
5.	$\text{wise}(c)$	$\rightarrow$-elim: 4,3
6.	$\exists x\,\text{wise}(x)$	$\exists$-intro: 5
7.	$\exists x\,\text{wise}(x)$	$\exists$-elim: 2,3-6

Yet Another Proof

Premise: $\forall x\,p(x) \wedge \forall x\,q(x)$
Conclusion: $\forall x\,(p(x) \wedge q(x))$

1.	$\forall x\,p(x) \wedge \forall x\,q(x)$	premise
2.	$\forall x\,p(x)$	$\wedge$-elim-1: 1
3.	$\forall x\,q(x)$	$\wedge$-elim-2: 1
	(Let a be an arbitrary element.)	
4.	$p(a)$	$\forall$-elim: 2
5.	$q(a)$	$\forall$-elim: 3
6.	$p(a) \wedge q(a)$	$\wedge$-intro: 3,4
7.	$\forall x\,(p(x) \wedge q(x))$	$\forall$-intro: 4-6

An Incorrect Proof

Premise: $\forall x \exists y\, r(x,y)$
Conclusion: $\exists y \forall x\, r(x,y)$

1.	$\forall x \exists y\, r(x,y)$	premise
2.	$\exists y\, r(c,y)$	$\forall$-elim: 1
3.	$r(c,d)$	$\exists$-elim: 2
4.	$\forall x\, r(x,d)$	$\forall$-intro: 3
5.	$\exists y \forall x\, r(x,y)$	$\exists$-intro: 4

One Last Proof

1.	$\forall x\, p(x)$	assumption
2.	$\exists x\, \neg p(x)$	assumption
	(Let a be the object satisfying 2.).	
3.	$\neg p(a)$	assumption
4.	$p(a)$	$\forall$-elim: 1
5.	*false*	*false*-intro: 4,3
6.	*false*	$\exists$-elim: 2,3-5
7.	$\neg\exists x\, \neg p(x)$	$\neg$-intro: 3-6
8.	$\forall x\, p(x) \rightarrow \neg\exists x\, \neg p(x)$	$\rightarrow$-intro: 2-8

A Correct Proof

1.	$\exists y \forall x\, r(x,y)$	premise
	(Let c be the object satisfying 1.)	
2.	$\forall x\, r(x,c)$	assumption
	(Let d be an arbitrary element.)	
3.	$r(d,c)$	$\forall$-elim: 2
4.	$\exists y\, r(d,y)$	$\exists$-intro: 3
5.	$\forall x \exists y\, r(x,y)$	$\forall$-intro: 3-4
6.	$\forall x \exists y\, r(x,y)$	$\exists$-elim: 1,2-5

Reading

Huth & Ryan:
Section 2.3
pp. 107–122.