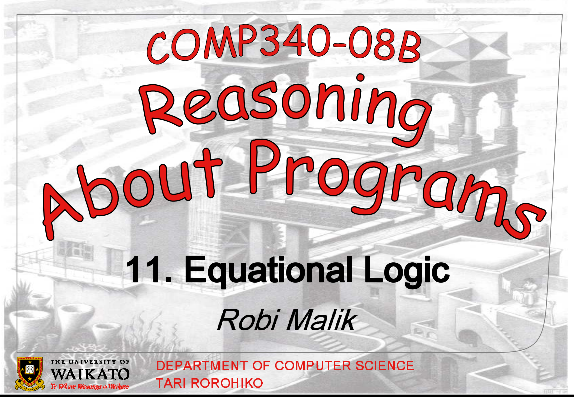

COMP340-08B Reasoning About Programs

11. Equational Logic

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Why Equality?

$$\text{superman} = \text{clark\_kent}$$

- Natural and concise notation
- Common in mathematical reasoning
- Adds some expressive power to predicate logic

The Substitution Principle

$$\text{superman} = \text{clark\_kent}$$

- If Superman and Clark Kent are the same person, then every property of Superman also is a property of Clark Kent.
- Thus, if Superman can fly, then Clark Kent can also fly, etc.

Natural Deduction Rules for Equality

=-Introduction (Reflexivity Axiom)

$$(\text{=-intro})\quad \frac{}{\forall x\ \ x = x}$$

=-Elimination (Substitution Principle)

$$(\text{=-elim})\quad \frac{A \qquad t_1 = t_2}{A[t_1/t_2]^k}$$

Substitutions

$A[t_1/t_2]^k$ is the result of replacing the *k*th occurrence of subterm t_1 in formula A by t_2.

$A[t_1/t_2]$ is the result of replacing all occurrences of subterm t_1 in formula A by t_2.

An Example Proof

(Let a be an arbitrary object.)

(Let b be an arbitrary object.)

1.	$a = b$	assumption
2.	$\forall x\ \ x = x$	=-intro
3.	$a = a$	∀-elim: 2
4.	$b = a$	=-elim: 3,1
5.	$a = b \rightarrow b = a$	→-intro: 1-4
6.	$\forall y\ (a = y \rightarrow y = a)$	∀-intro: 1-5
7.	$\forall x \forall y\ (x = y \rightarrow y = x)$	∀-intro: 1-6

Another Example

	(Let a be an arbitrary object.)	
	(Let b be an arbitrary object.)	
	(Let c be an arbitrary object.)	
1.	$a = b \wedge b = c$	assumption
2.	$a = b$	∧-elim-1: 1
3.	$b = c$	∧-elim-2: 1
4.	$a = c$	=-elim: 2,3
5.	$a = b \wedge b = c \rightarrow a = c$	→-intro: 1-4
6.	$\forall z\ (a = b \wedge b = z \rightarrow a = z)$	∀-intro: 1-5
7.	$\forall y\ \forall z\ (a = y \wedge y = z \rightarrow a = z)$	∀-intro: 1-6
8.	$\forall x\ \forall y\ \forall z\ (x = y \wedge y = z \rightarrow x = z)$	∀-intro: 1-7

Properties of Equality

- Reflexivity

$$\forall x\ \ x = x$$

- Symmetry

$$\forall x\, \forall y\ (x = y \rightarrow y = x)$$

- Transitivity

$$\forall x\, \forall y\, \forall z\ (x = y \wedge y = z \rightarrow x = z)$$

Yet Another Example

1.	$a = b$	premise
2.	$p(a)$	assumption
3.	$p(b)$	=-elim: 2,1
4.	$p(a) \rightarrow p(b)$	→-intro: 2–3
5.	$p(b)$	assumption
6.	$\forall x\ \ x = x$	=-intro
7.	$a = a$	∀-elim: 6
8.	$b = a$	=-elim: 7,1
9.	$p(a)$	=-elim: 5,8
10.	$p(a) \rightarrow p(b)$	→-intro: 5–9
11.	$p(a) \leftrightarrow p(b)$	↔-intro: 4,10

Equality Trap

What is wrong with this formalisation?

- The lion is a mammal.

$$\text{lion} = \text{mammal}$$

Expressing Uniqueness

- H.C. is the only prime minister of New Zealand.

$$\text{pm(hc)} \wedge \forall x\ (\text{pm}(x) \rightarrow x = \text{hc})$$

- There is one and only one prime minister of New Zealand.

$$\exists x\ (\text{pm}(x) \wedge \forall y\ (\text{pm}(y) \rightarrow x = y))$$

Unique Existence Quantifier

$$\exists_1 x\ A(x)$$

is often used as a shorthand for

$$\exists x\ (A(x) \wedge \forall y\ (A(y) \rightarrow x = y))$$