COMP340-08B

Reasoning About Programs

16. Soundness & Completeness

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Natural Deduction

A simple deductive system
for propositional and predicate logic.

- 10 rules for connectives $\wedge$ $\vee$ $\rightarrow$ $\leftrightarrow$
- 3 rules for $\neg$
- 4 rules for quantifiers $\forall$ $\exists$

Are these rules sufficient to prove everything we want to prove?

The Deductive System D1

Axioms:

(**A1**) $A \rightarrow (B \rightarrow A)$

(**A2**) $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))$

(**A3**) $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$

(**A4**) $\neg\neg A \rightarrow A$

Rules of inference:

(**MP**) $$\frac{A \quad A \rightarrow B}{B}$$

The Deductive System R1

Axioms:

(none)

Rules of inference:

(**Res**) $$\frac{A \vee C \quad B \vee \neg C}{A \vee B}$$

Deductive Systems

A deductive system consists of a set of axioms and rules.

Questions

- Which axioms and rules should we chose?
- When does a deductive system make sense?

Syntax versus Semantics

Semantics

- A formula A is **valid** if it is true for all possible interpretation.

Syntax

- A formula A is **provable** in a deductive system S if there exists a **proof** for the formula A using only the axioms and rules of S.

When are these two concepts the same?

What is a Proof?

Definition:

Let S be a deductive system. A **derivation** from in S is a finite sequence $\mathcal{D} = (A_1,\ldots,A_n)$ of formulas A_i such that, for each $i = 1,\ldots,n$, either

- A_i is obtained from an axiom of S, or
- A_i is obtained by applying a rule of S to some formulas occurring before A_i in $\mathcal{D}$.

What is a Proof?

Definition:

Let S be a deductive system, and let A be a formula.

- A is called **provable** in S, if there exists a derivation $\mathcal{D} = (A_1,\ldots,A_n)$ in S such that $A_n = A$.
- Then $\mathcal{D}$ is called a **proof** for A is S.

Soundness and Completeness

Definition:

Let S be a deductive system.

- S is called **sound** if every formula that is provable in S is valid.
- S is called **complete** if every valid formula is provable in S.

Some Good Deductive Systems

Theorem:

The deductive system of natural deduction is sound and complete.

Theorem:

The deductive system **D1** is sound and complete.

Proving that Axioms are Valid

A	B	$B \to A$	**A1**	$A \to B$	$A \to \neg B$	$(A \to \neg B) \to \neg A$	**A3**	$\neg A$	$\neg\neg A$	**A4**
F	F	T	T	T	T	T	T	T	F	T
F	T	F	T	T	T	T	T	T	F	T
T	F	T	T	F	T	F	T	F	T	T
T	T	T	T	T	F	T	T	F	T	T

A	B	C	$A \to B$	$B \to C$	$A \to (B \to C)$	$A \to C$	$(A \to (B \to C)) \to (A \to C)$	**A2**
F	F	F	T	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	T	F	T	F	T	T	T	T
F	T	T	T	T	T	T	T	T
T	F	F	F	T	T	F	F	T
T	F	T	F	T	T	T	T	T
T	T	F	T	F	F	F	T	T
T	T	T	T	T	T	T	T	T

How to Prove Soundness

Theorem:

Let S be a deductive system such that

- every axiom of S is a valid formula, and
- for every rule in S

$$\frac{A_1 \ldots A_n}{B}$$

B is a logical consequence of $A_1,\ldots,A_n$.

Then S is sound.