COMP340-08B Reasoning About Programs

18. Completeness of Resolution

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Resolution

A deductive system for clauses with no axioms and only one rule.

$$\text{(Res)} \quad \frac{C_1 \cup \{p\} \qquad C_2 \cup \{\neg p\}}{C_1 \cup C_2}$$

Completeness of Resolution

Resolution is refutation-complete.

Theorem

Let Ψ be a set of clauses.
If Ψ is **unsatisfiable**, then the empty clause $\square$ can be proven from Ψ by resolution.

Some Notation

Definition

Let Ψ be a set of clauses.
$\textsc{Res}(\Psi)$ denotes the set of all clauses that can be obtained by resolution from Ψ.

Examples

$$\Psi = \{\{p, \neg q\}, \{\neg p\}\}$$
$$\textsc{Res}(\Psi) =$$

$$\Psi = \{\{p, \neg q\}, \{\neg p, r\}, \{q\}, \{\neg r\}\}$$
$$\textsc{Res}(\Psi) =$$

Restating the Completeness Theorem

Resolution is refutation-complete.

Theorem

Let Ψ be a set of clauses.
If Ψ is unsatisfiable, then $\square \in \textsc{Res}(\Psi)$.

Proving the Completeness Theorem

Proof

Let Ψ be an unsatisfiable set of clauses.

We prove that $\Box \in \text{Res}(\Psi)$ by induction on the number n of propositional variables in Ψ.

Inductive Base

Inductive Base.

Let $n = 0$, i.e., Ψ contains no propositional variables. There are only two possible sets of clauses without propositional variables:

- $\Psi = \{\}$ is satisfiable, and therefore does not satisfy the assumption.
- $\Psi = \{\Box\}$ contains the empty clause $\Box$, and therefore $\Box \in \Psi \subseteq \text{Res}(\Psi)$.

Inductive Hypothesis

Inductive Hypothesis.

For every unsatisfiable set Ψ of clauses with n propositional variables, it holds that $\Box \in \text{Res}(\Psi)$.

Or, equivalently, every set Ψ of clauses with n propositional variables such that $\Box \notin \text{Res}(\Psi)$ is satisfiable.

Inductive Step

Inductive Step. $n \to n + 1$

We prove the inductive step using **indirect proof**.
So let Ψ be a set of clauses with $n + 1$ propositional variables such that $\Box \notin \text{Res}(\Psi)$.
We will show that Ψ is **satisfiable**.

Let p be one of the propositional variables in Ψ.

Then $\{\neg \mathrm{p}\} \notin \text{Res}(\Psi)$ or $\{\mathrm{p}\} \notin \text{Res}(\Psi)$,
because otherwise $\Box \in \text{Res}(\Psi)$ in contradiction to the assumption.

So we consider two cases.

Inductive Step — Case 1

Case 1. $\{\neg \mathrm{p}\} \notin \text{Res}(\Psi)$

Define the set of clauses Ψ^{p} as follows,

$$\Psi^{\mathrm{p}} = \{\, C \setminus \{\neg \mathrm{p}\} \mid C \in \Psi \text{ and } \mathrm{p} \notin C \,\} .$$

Note that

$$\text{Res}(\Psi^{\mathrm{p}}) \subseteq \{\, C \setminus \{\neg \mathrm{p}\} \mid C \in \text{Res}(\Psi) \text{ and } \mathrm{p} \notin C \,\}$$

and therefore

$$\Box \notin \text{Res}(\Psi^{\mathrm{p}})$$

because $\Box \notin \text{Res}(\Psi)$ and $\{\neg \mathrm{p}\} \notin \text{Res}(\Psi)$.

By inductive hypothesis, Ψ^{p} is **satisfiable**.

Inductive Step — the End

$$\Psi^{\mathrm{p}} = \{\, C \setminus \{\neg \mathrm{p}\} \mid C \in \Psi \text{ and } \mathrm{p} \notin C \,\} .$$

Since Ψ^{p} is **satisfiable**, it has a **model**, say I^{p}.

I^{p} assigns truth values to all propositional variables in Ψ except p.

Construct an interpretation I as follows.

$$I(x) = \begin{cases} \mathbf{T}, & \text{if } x = \mathrm{p}; \\ I^{\mathrm{p}}(x), & \text{otherwise.} \end{cases}$$

Then I is a model for Ψ, i.e., Ψ is satisfiable.

Case 2. $\{\mathrm{p}\} \notin \text{Res}(\Psi)$ is analogous. $\Box$