COMP340-08B Reasoning About Programs

35. Formalising Set Theory

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Signature for Sets

- New function symbols
  - $\mathbf{U}$ $\varnothing$ (constant symbols)
  - $\cup$ $\cap$ $\setminus$ (binary function symbols)
- New predicate symbols
  - $\in$ $\notin$ $\subseteq$ (binary predicate symbols)
- Written in infix notation for convenience
  - Write $A \cup B$ instead of $\cup(A, B)$.

Formalising Properties of Sets

(univ)	$\forall x\,(x \in \mathbf{U})$
(empty)	$\neg\exists x\,(x \in \varnothing)$
(inters)	$\forall X\,\forall Y\,\forall x\,(x \in X \cap Y \leftrightarrow x \in X \land x \in Y)$
(union)	$\forall X\,\forall Y\,\forall x\,(x \in X \cup Y \leftrightarrow x \in X \lor x \in Y)$
(setdiff)	$\forall X\,\forall Y\,\forall x\,(x \in X \setminus Y \leftrightarrow x \in X \land x \notin Y)$
(notin)	$\forall X\,\forall x\,(x \notin X \leftrightarrow \neg(x \in X))$
(subset)	$\forall X\,\forall Y\,(X \subseteq Y \leftrightarrow \forall x\,(x \in X \rightarrow x \in Y))$
(eqset)	$\forall X\,\forall Y\,(X = Y \leftrightarrow X \subseteq Y \land X \subseteq Y)$

A Proof in Set Theory

1.	$\forall X\,\forall Y\,\forall x\,(x \in X{\cup}Y \leftrightarrow x{\in}X \lor x{\in}Y)$	Axiom (union)
2.	$\forall X\,\forall Y\,(X \subseteq Y \leftrightarrow \forall x\,(x{\in}X \rightarrow x{\in}Y))$	Axiom (subset)
	(Let A, B be arbitrary elements.)	
	(Let c be an arbitrary element.)	
3.	$c \in A$	Assumption
4.	$c \in A \lor c \in B$	$\lor$-intro-1: 3
5.	$c \in A \cup B \leftrightarrow c \in A \lor c \in B$	$\forall$-elim: 1
6.	$c \in A \lor c \in B \rightarrow c \in A \cup B$	$\leftrightarrow$-elim-2: 5
7.	$c \in A \cup B$	$\rightarrow$-elim: 4,6
8.	$c \in A \rightarrow c{\in}\ A \cup B$	$\rightarrow$-intro: 3-7
9.	$\forall x\,(x \in A \rightarrow x{\in}\ A \cup B)$	$\forall$-intro: 3-8

Set Theory Proof continued

1.	$\forall X\,\forall Y\,\forall x\,(x \in X{\cup}Y \leftrightarrow x{\in}X \lor x{\in}Y)$	Axiom (union)
2.	$\forall X\,\forall Y\,(X \subseteq Y \leftrightarrow \forall x\,(x{\in}X \rightarrow x{\in}Y))$	Axiom (subset)
	(Let A, B be arbitrary elements.)	
	…	
9.	$\forall x\,(x \in A \rightarrow x{\in}\ A \cup B)$	$\forall$-intro: 3-8
10.	$A \subseteq A \cup B \leftrightarrow \forall x\,(x \in A \rightarrow x \in A \cup B)$	$\forall$-elim: 2
11.	$\forall x\,(x \in A \rightarrow x \in A \cup B) \rightarrow A \subseteq A \cup B$	$\leftrightarrow$-elim-2: 10
12.	$A \subseteq A \cup B$	$\rightarrow$-elim: 10,11
13.	$\forall X\,\forall Y\,(X \subseteq X \cup Y)$	$\forall$-intro: 3-12

More: Set Comprehension

- **The set of all white elephants …**

$$\mathbf{WE} = \{\, x \mid \text{elephant}(x) \land \text{white}(x) \,\}$$

- **The set of all prime numbers …**

$$\mathbf{Primes} = \{\, x \mid \forall d\,(d \mid x \rightarrow d = 1 \lor d = x) \,\}$$

- **Defining set operators …**

$$A \cup B = \{\, x \mid x \in A \lor x \in B \,\}$$

$$A \setminus B = \{\, x \mid x \in A \land x \notin B \,\}$$

Formalising Set Comprehension

Additional Syntax

$$\{\ x \mid A(x)\ \}$$

where $A(x)$ is some predicate logic formula containing x as a free variable.

Set Comprehension Axiom

(set) $\forall y\ (y \in \{\ x \mid A\ \} \leftrightarrow A[x/y])$

Russell's Paradox

*Some sets, such as the set of all teacups, are not members of themselves. Other sets, such as the set of all non-teacups, are members of themselves. Call the set of all sets that are not members of themselves **R**. If **R** is a member of itself, then by definition it must not be a member of itself. Similarly, if **R** is not a member of itself, then by definition it must be a member of itself.*

(Discovered by Bertrand Russell in 1901)

The Russell Set

Consider the set

$$\{\ x \mid x \notin x\ \}$$

For convenience, we use the letter

$$\mathbf{R}$$

as a shorthand notation for the above set.

"Proving" the Paradox

1.	$\forall X\ \forall x\ (x \notin X \leftrightarrow \neg(x \in X))$	Axiom (notin)
2.	$\forall y\ (y \in \{\ x \mid x \notin x\ \} \leftrightarrow y \notin y)$	Axiom (set)
3.	$\mathbf{R} = \{\ x \mid x \notin x\ \}$	Definition of $\mathbf{R}$
4.	$\forall y\ (y \in \mathbf{R} \leftrightarrow y \notin y)$	=-elim: 2,3
5.	$\mathbf{R} \in \mathbf{R} \leftrightarrow \mathbf{R} \notin \mathbf{R}$	∀-elim: 4
6.	$\mathbf{R} \notin \mathbf{R} \leftrightarrow \neg(\mathbf{R} \in \mathbf{R})$	∀-elim: 1
...	…	propositional logic
20.	false	

The Consequences

The Problem

- The naive form of set comprehension makes natural deduction unsound.

A Solution: Type Theory

- Everything must be strictly typed.
- We can have elements, sets of elements, sets of sets of elements, etc.
- <u>But:</u> We cannot have arbitrary sets.

Typed Examples

$\forall x{:}\mathbb{N}\ \forall y{:}\mathbb{N}\ (x \mid y \leftrightarrow \exists n{:}\mathbb{N}\ n \cdot x = y)$

Primes: $\mathbb{PN}$

Primes $= \{\ x{:}\mathbb{N} \mid \forall d{:}\mathbb{N}\ (d \mid x \rightarrow d = 1 \vee d = x)\ \}$

WE: $\mathbb{P}\mathbf{A}$

WE $= \{\ x{:}\mathbf{A} \mid \text{elephant}(x) \wedge \text{white}(x)\ \}$

$\cup$: $\mathbb{P}\mathbf{D} \times \mathbb{P}\mathbf{D} \rightarrow \mathbb{P}\mathbf{D}$

$\forall X{:}\mathbb{P}\mathbf{D}\ \forall Y{:}\mathbb{P}\mathbf{D}\ \ X \cup Y = \{\ x{:}\mathbf{D} \mid x \in X \vee x \in Y\ \}$